Symmetric and conserved energy-momentum tensors in moving media

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- Photon momenta in media
- Minkowski and Abraham tensors
- A scalar field in the Gordon metric
- The new energy-momentum tensor
- Conclusion

Photon in medium with refractive index n:

Energy $E = h\nu$ and momentum $p = h/\lambda$

with $\nu\lambda = c = 1/n$

i.e. $E = p/n$ Minkowski momentum

Abraham momentum: $E = np$ Quantum mechanics??

Photon mass

Minkowski: $E^2 - p^2 = (1 - n^2)E^2 < 0$ tachyon!

Abraham: $E^2 - p^2 = (n^2 - 1)p^2 > 0$

Maxwell equations:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0, \qquad \nabla \cdot \mathbf{D} = 0$$

Constitutive equations $\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$

with index of refraction $n = \sqrt{\varepsilon \mu}$

Energy density $\mathcal{E} = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$

and Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$

Energy conservation: $\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{S} = 0$

Momentum density $\mathbf{G} = \mathbf{D} \times \mathbf{B}$

and conservation $\dfrac{\partial \mathbf{G}}{\partial t} + \nabla \cdot \mathbf{T} = 0$

where $T_{ij} = -(E_i D_j + B_i H_j) + \frac{1}{2}\delta_{ij}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$

are Maxwell stresses.

Covariant formulation

Minkowski energy-momentum tensor: $\mathbf{D} \times \mathbf{B} = n^2 \mathbf{E} \times \mathbf{H}$

$$T_M^{\mu\nu} = \left(\begin{array}{c|c} \mathcal{E} & \mathbf{E} \times \mathbf{H} \\ \hline \mathbf{D} \times \mathbf{B} & T_{ij} \end{array}\right)$$

Abraham energy-momentum tensor:

$$T_A^{\mu\nu} = \left(\begin{array}{c|c} \mathcal{E} & \mathbf{E} \times \mathbf{H} \\ \hline \mathbf{E} \times \mathbf{H} & T_{ij} \end{array}\right)$$

Electromagnetic Lagrangian in dielectrikum:

$$\mathcal{L} = \frac{1}{2} n^2 \mathbf{E}^2 - \frac{1}{2} \mathbf{B}^2$$

with $\mathbf{E} = -\partial \mathbf{A} / \partial t$ and $\mathbf{B} = \nabla \times \mathbf{A}$

Instead consider scalar, massless field:

$$\mathcal{L} = \frac{1}{2} n^2 \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} (\nabla \phi)^2$$

Can now make covariant formulation by considering field in external metric:

$$\hat{\eta}^{\mu\nu} = (n^2, -1, -1, -1) \rightarrow \eta^{\mu\nu} + (n^2 - 1) u^\mu u^\nu$$

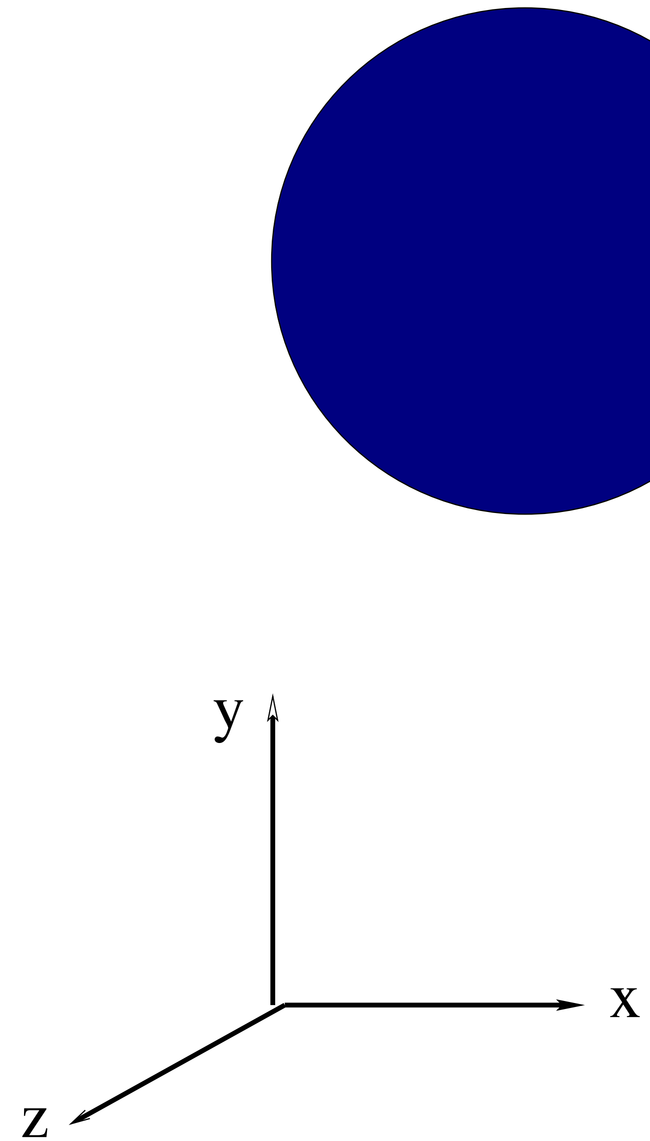
$$\Longrightarrow \qquad \mathcal{L} = (1/2) \hat{\eta}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

Equation of motion: $\hat{\eta}^{\mu\nu} \partial_\mu \partial_\nu \phi = [\partial^\mu \partial_\mu + (n^2 - 1)(u^\mu \partial_\mu)^2] \phi = 0$

Gordon, 1923: $\hat{\eta}^{\mu\nu} \to \hat{g}^{\mu\nu}(x)$

Action $S[\phi] = \frac{1}{2} \int d^4x \sqrt{-\hat{g}}\, \hat{g}^{\mu\nu} \partial_\mu \phi\, \partial_\nu \phi$

Equivalent gravitational problem



Energy-momentum tensor from

$$\hat{g}_{\mu\nu} \to \hat{g}_{\mu\nu} + \delta\hat{g}_{\mu\nu}$$

Using $\quad \delta\hat{g}^{\mu\nu} = -\hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}\delta\hat{g}_{\alpha\beta}$

and $\quad \delta\sqrt{-\hat{g}} = (1/2)\sqrt{-\hat{g}}\hat{g}^{\alpha\beta}\delta\hat{g}_{\alpha\beta}$

$$\Longrightarrow \quad \delta S[\phi] = -\frac{1}{2}\int d^4x\sqrt{-\hat{g}}\,[\hat{g}^{\alpha\mu}\hat{g}^{\beta\nu}\partial_\mu\phi\,\partial_\nu\phi - \hat{g}^{\alpha\beta}\mathcal{L}]\delta\hat{g}_{\alpha\beta}$$

with the result:

$$\boxed{\hat{T}^{\mu\nu} = \hat{\eta}^{\mu\alpha}\hat{\eta}^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi - \hat{\eta}^{\mu\nu}\mathcal{L}}$$

Symmetric: $\quad \hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$

Conserved: $\quad \partial_\mu\hat{T}^{\mu\nu} = 0$

Electromagnetic fields

Field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \left(\begin{array}{c|c} 0 & \mathbf{E} \\ \hline -\mathbf{E} & -B_{ij} \end{array}\right)$$

Equation of motion: $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$

Material fields

$$H^{\mu\nu} = \hat{\eta}^{\mu\alpha}\hat{\eta}^{\nu\beta}F_{\alpha\beta}$$

$$= \left(\begin{array}{c|c} 0 & -\mathbf{D} \\ \hline \mathbf{D} & -H_{ij} \end{array}\right)$$

Equation of motion:

$$\partial_\mu H^{\mu\nu} = 0$$

Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}H^{\mu\nu}$$

Energy-momentum tensor derived by same procedure

$$\Longrightarrow \qquad \boxed{\hat{T}^{\mu\nu} = \hat{\eta}^{\mu\alpha} F_{\alpha\beta} H^{\beta\nu} - \hat{\eta}^{\mu\nu} \mathcal{L}}$$

Symmetric $\qquad \hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$

and conserved $\qquad \partial_\mu \hat{T}^{\mu\nu} = 0$

Minkowski energy-momentum tensor:

$$T^{\mu\nu} = \eta^{\mu\alpha} F_{\alpha\beta} H^{\beta\nu} - \eta^{\mu\nu} \mathcal{L}$$

In medium rest frame:

$$\hat{T}^{\mu\nu} = \left(\begin{array}{c|c} n^2\mathcal{E} & \mathbf{D}\times\mathbf{B} \\ \hline \mathbf{D}\times\mathbf{B} & \hat{T}_{ij} \end{array}\right)$$

$$\mathbf{D}\times\mathbf{B} = n^2\mathbf{E}\times\mathbf{H}$$

Extra factor in energy components from red shift $\hat{g}_{00} = \dfrac{1}{n^2}$

NB!

There is another energy-momentum tensor that couples to the metric $g_{\mu\nu}$

This turns out to be exactly the Abraham tensor

that describes the coupled field-medium system!

F. Ravndal: arXiv: 1107.5074