Scalars in Conformal Field Theories

Slava Rychkov

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Many reasons to think about CFT’s

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UV structure Beyond the Standard Model
- ad hoc CFT hidden sectors (unparticles)
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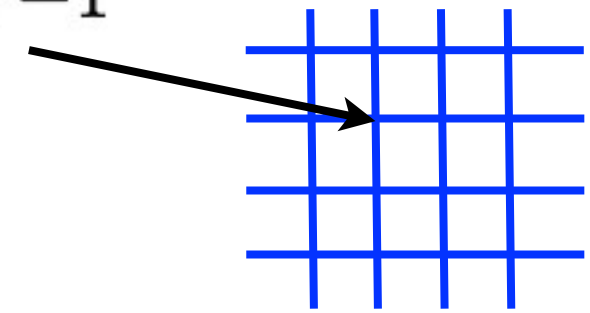
Critical phenomena
- statistical mechanics
- quantum condensed matter (=quantum criticality)

This talk

Classical O(N)-invariant ferromagnet in d=3

$$Z = \exp\left[-\frac{1}{T}\sum_{\langle ij\rangle}\sigma_i\cdot\sigma_j\right]$$

$$\sigma_i \in S^{N-1}$$



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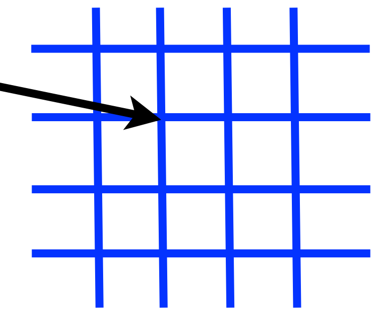

N=1: Ising model **($\sigma = \pm 1$)**

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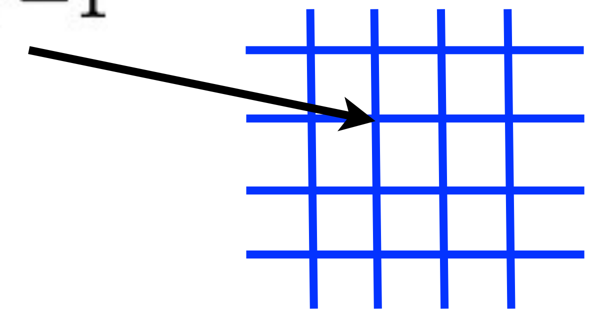

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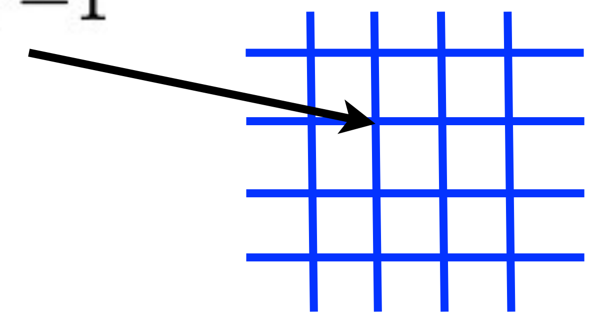

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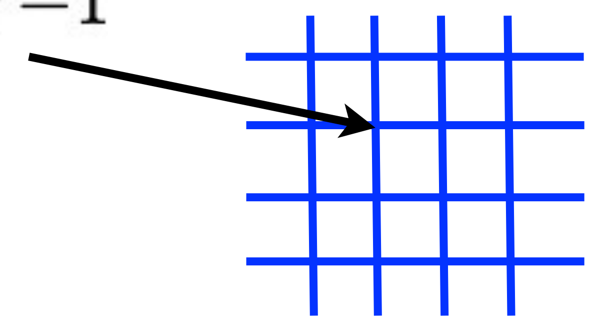

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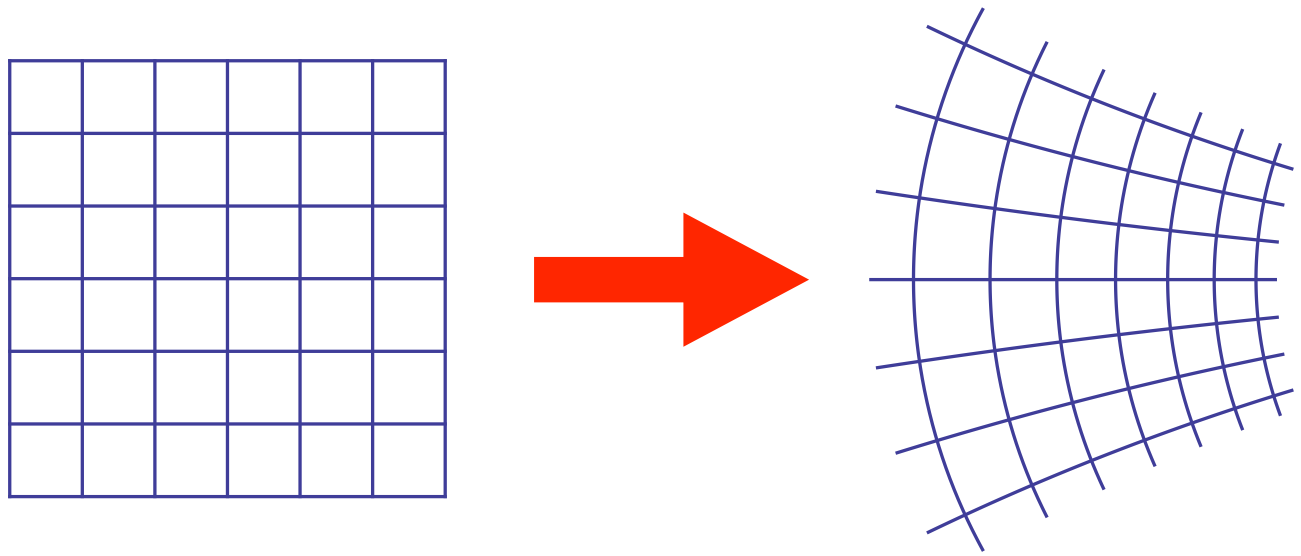
At $T=T_c$ theory is 1) scale invariant
2) conformal

Special Conformal transformation

$$\delta_\kappa x_a = 2(\kappa \cdot x)x_a - x^2\kappa_a$$



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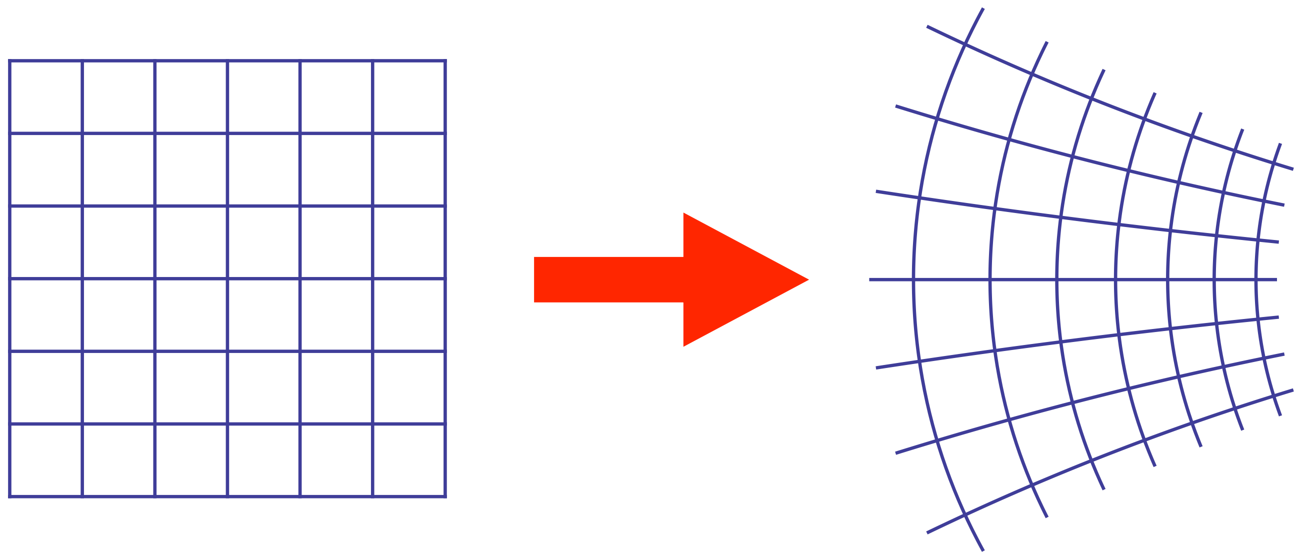

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Are there interesting scale invariant theories without full conformal invariance?

Polchinski 1988, Dorigoni, S.R. 2009,
El-Showk, Nakayama, S.R. 2011, Antoniadis, Buican 2011
Fortin, Grinstein, Stergiou 2011

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Related to the dimension of another local field, ε(x):

$$\nu = \frac{1}{d-\Delta_\epsilon}$$

Energy density field, ε(x)

$\varepsilon \sim$ "σ^2"

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Lagrangian describing the near-critical system:

$$\mathcal{L}_{CFT} + \frac{T/T_c - 1}{a^{d-\Delta_\epsilon}}\,\epsilon(x)$$

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$$\underbrace{}_{\sim \frac{1}{\xi(T)^{d-\Delta_\epsilon}}}$$

Correlation length develops at the length scale where perturbation grows to O(1) in strength $\Longrightarrow \quad \nu = \dfrac{1}{d-\Delta_\epsilon}$

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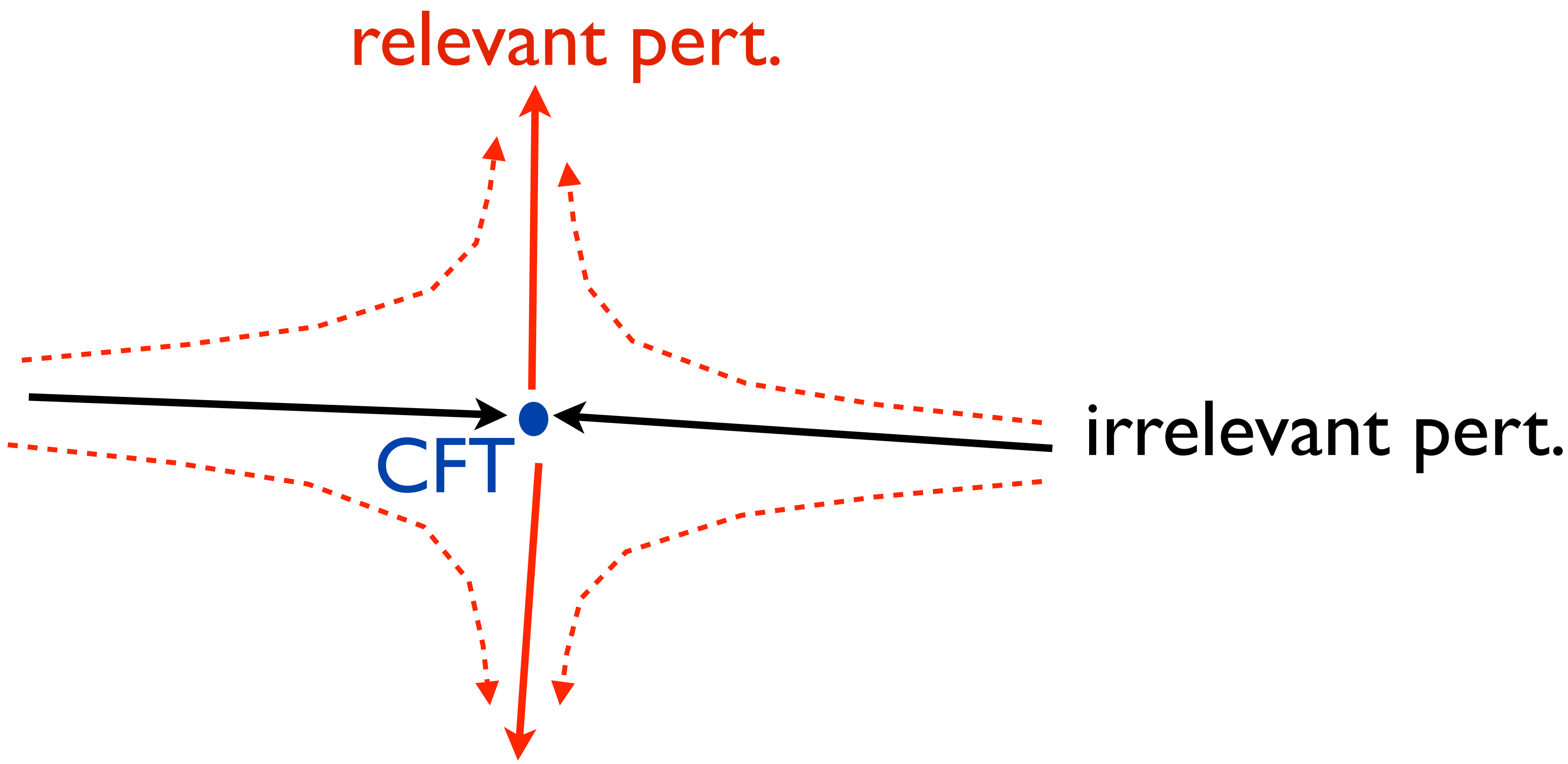

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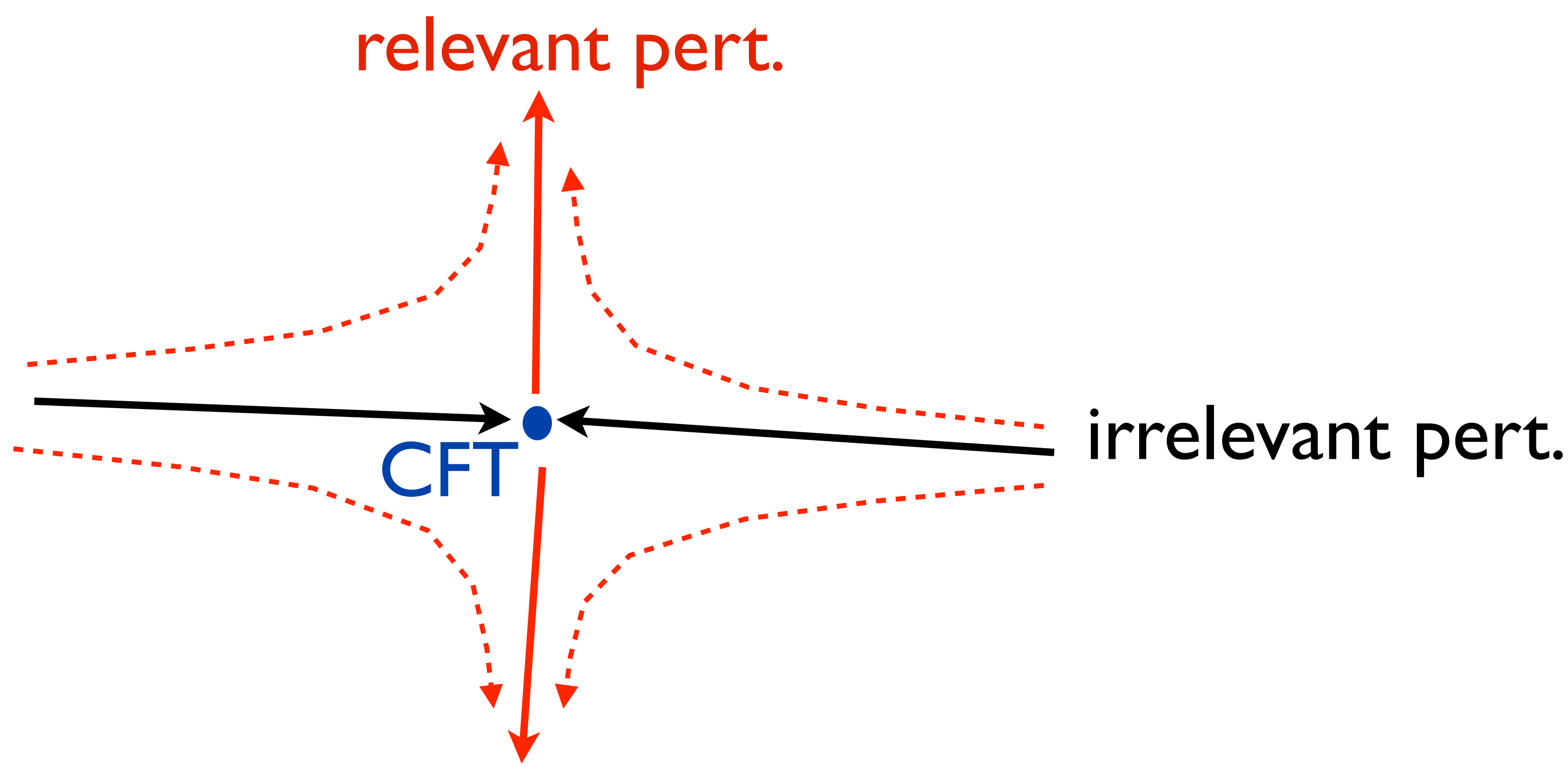
In particular from $\nu>0$ it follows $\Delta_\varepsilon<d$ (relevant operator)



Important: ε is **the only** relevant operator which is singlet under Z_2 symmetry $\boldsymbol{\sigma} \to \boldsymbol{-\sigma}$ (otherwise multicriticality)

How to determine critical exponents

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- Field theory techniques in 4-ε dimensions

ε-expansion

Wilson,Fischer

Study scalar field theory in 4-ε dimensions

$$\mathscr{L} = (\partial\phi)^2 + \lambda\phi^4$$

$$\beta_\lambda = -\epsilon\lambda + \frac{\lambda^2}{16\pi^2} + \ldots \to 0$$

$$\frac{\lambda_*}{16\pi^2} = O(\epsilon) \ll 1$$

weakly coupled fixed point

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Compute critical exponents at ε <<1 and then extrapolate to ε =1

Works pretty well but not to arbitrary accuracy (divergent series)

I think the epsilon expansion ended the subject in the practical sense.

You can calculate more or less what you want with good accuracy
but aesthetically the subject is not closed yet.

It's possible that there will be classification of fixed points in three dimensions.
But that's just dreams.

A.M. Polyakov, 2003 interview

Conformal bootstrap

Ferrara,Gatto,Grillo 1973
Polyakov 1974

Consistency eq. four 4pt function $\langle\sigma(x_1)\,\sigma(x_2)\,\sigma(x_3)\,\sigma(x_4)\rangle$

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$$\langle \sigma\, \sigma\, \epsilon \rangle \neq 0$$

Conformal symmetry fixes:

$$\langle \sigma(x)\, \sigma(y)\, \epsilon(0) \rangle = \frac{\lambda}{|x-y|^{2\Delta_\sigma - \Delta_\epsilon}\, |x|^{2\Delta_\epsilon}\, |y|^{2\Delta_\epsilon}}$$

Conformal OPE

$$\sigma(x_1)\,\sigma(x_2) = \sum_O \lambda_O\, C(x_1 - x_2, \partial_{x_2})\, O(x_2)$$

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$$O = O_\Delta^{(l)}$$

$$l = 2, 4, 6, \ldots$$

$$\Delta \geq l + d + 2 \quad (\geq d/2 - 1, l = 0)$$

$$\langle \underbrace{\sigma(x_1)\,\sigma(x_2)}_{\text{OPE}}\,\underbrace{\sigma(x_3)\,\sigma(x_4)}_{\text{OPE}} \rangle$$

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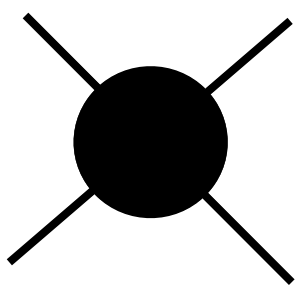
Conformal partial wave



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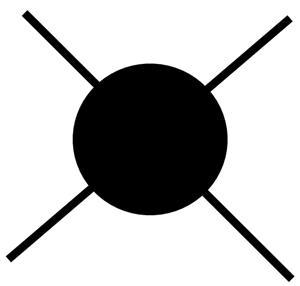

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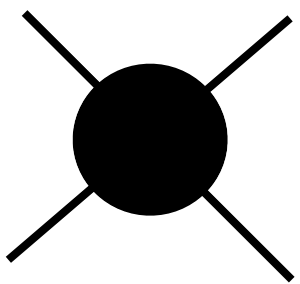

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$$g(u,v) = 1 + \sum_O \lambda_O^2\, g_O(u,v)$$

known functions of *u,v*

Crossing symmetry/OPE associativity

$$\langle \underbrace{\sigma(x_1)\,\sigma(x_2)}_{\text{OPE}}\,\underbrace{\sigma(x_3)\,\sigma(x_4)}_{\text{OPE}} \rangle \quad = \quad \langle \overbrace{\sigma(x_1)\,\underbrace{\sigma(x_2)\,\sigma(x_3)}_{\text{OPE}}}^{\text{OPE}}\,\sigma(x_4) \rangle$$

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Conformal bootstrap equation for CFT couplings and spectrum
(no progress for 30 years)

Resurrecting Conformal Bootstrap

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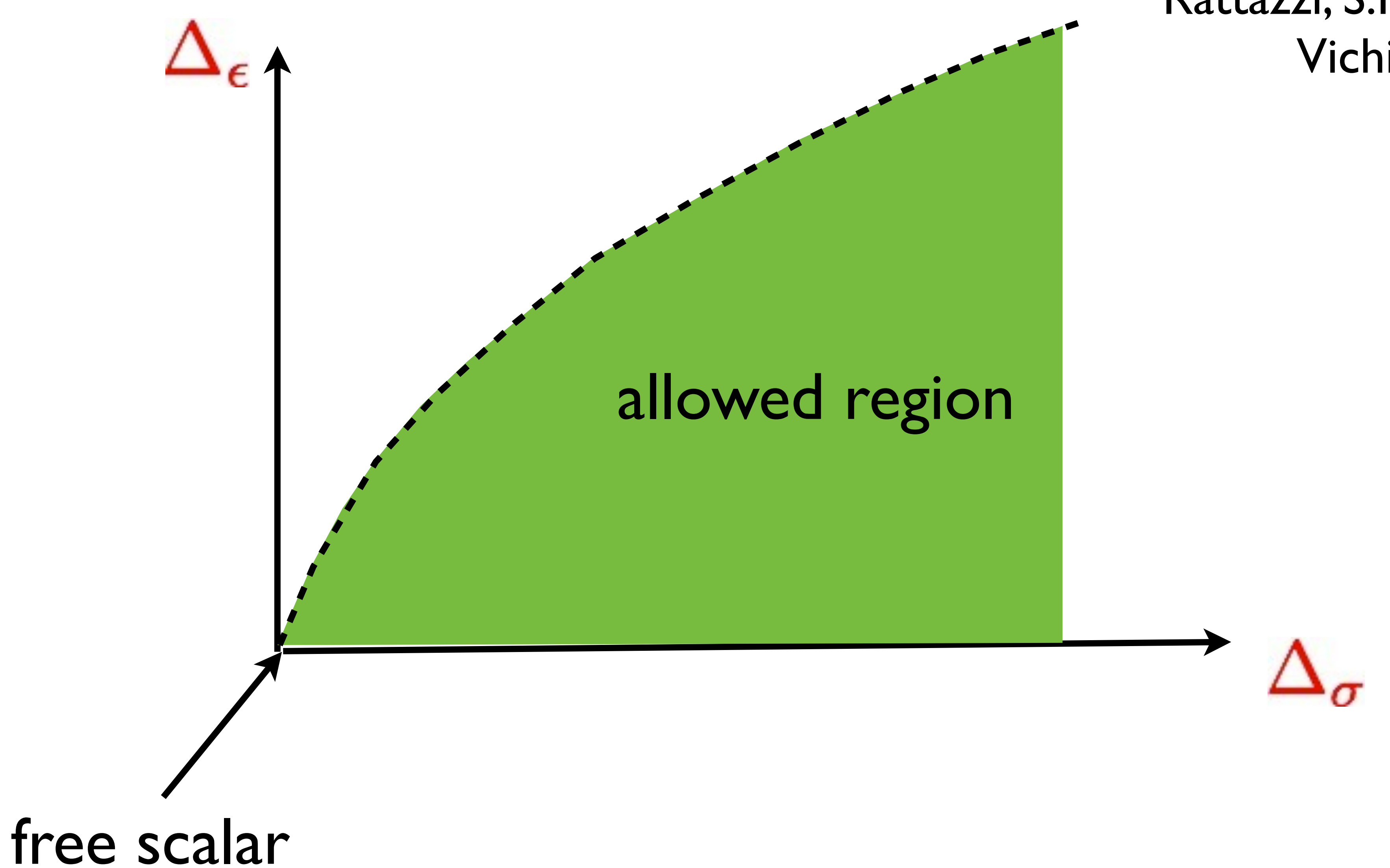

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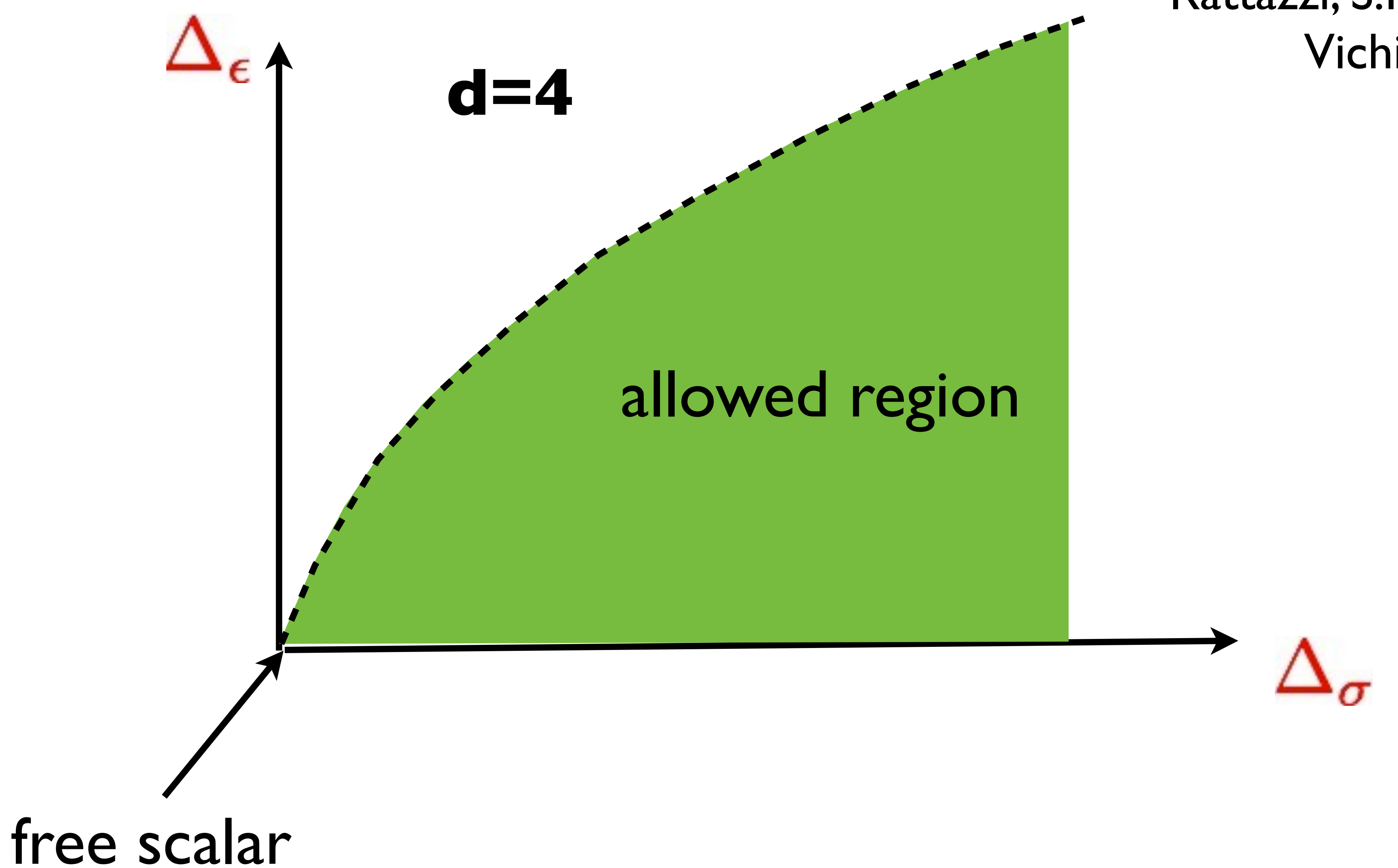
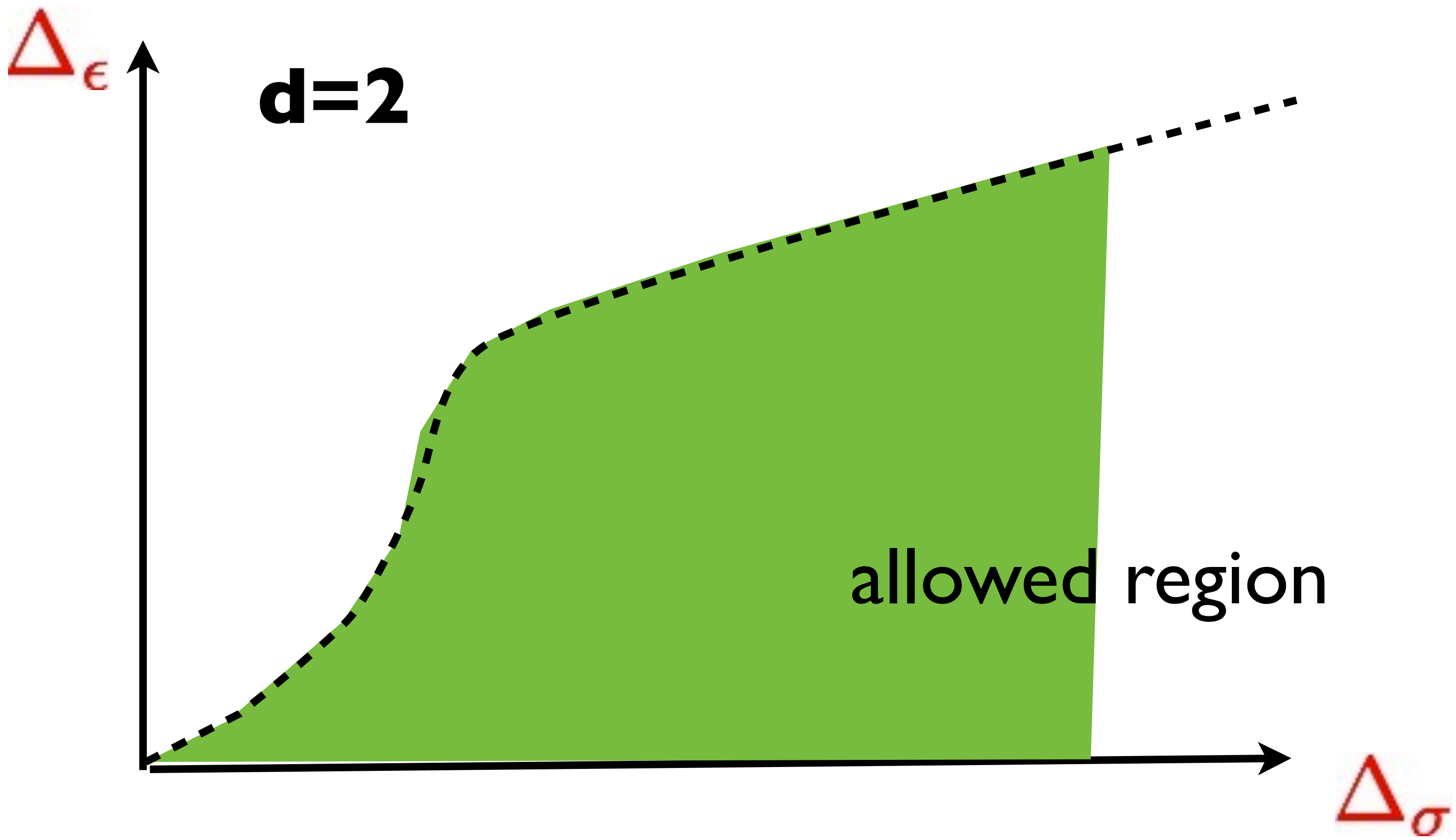


Motivated by Conformal Technicolor

Exercise in d=2

S.R., Vichi 2009



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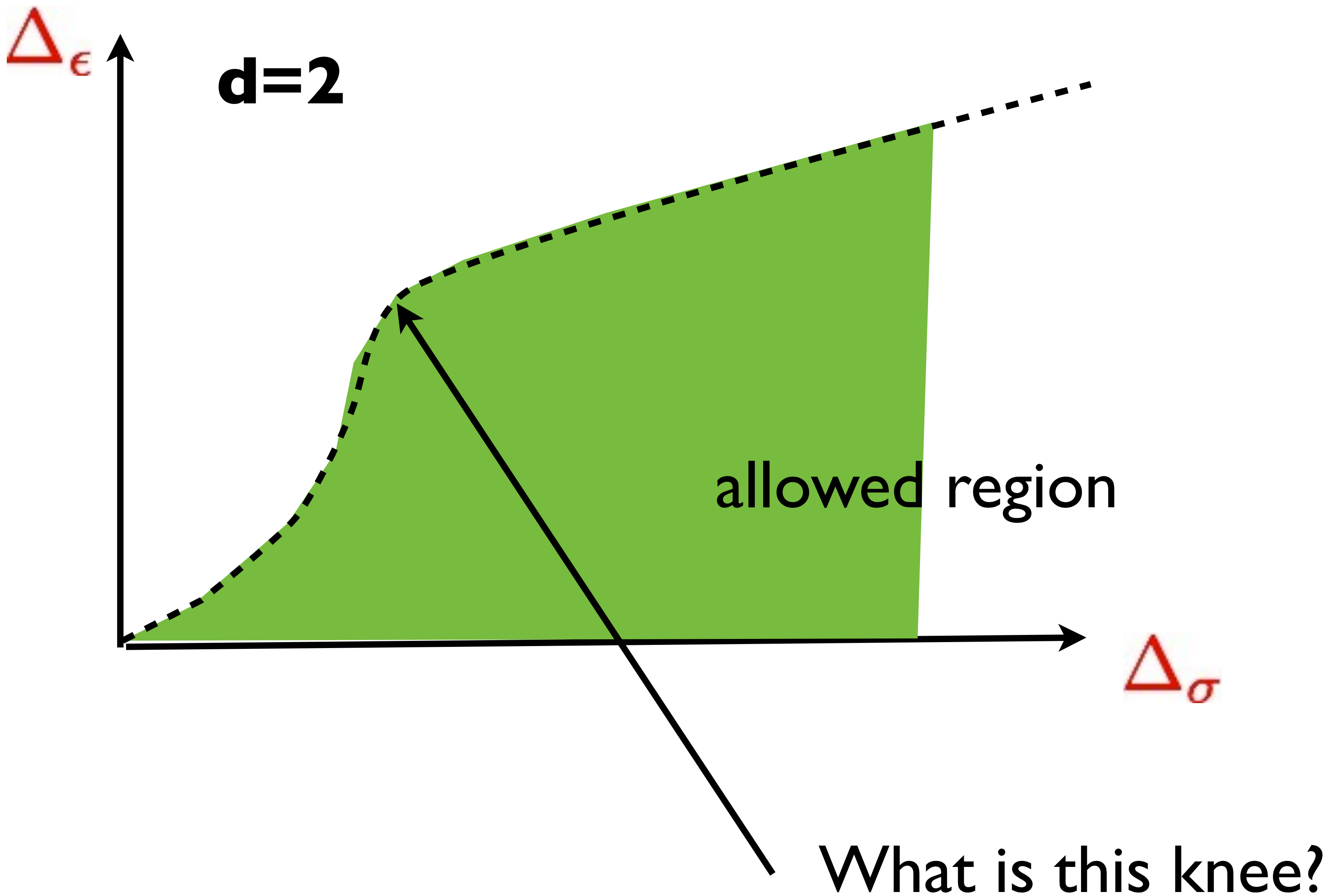

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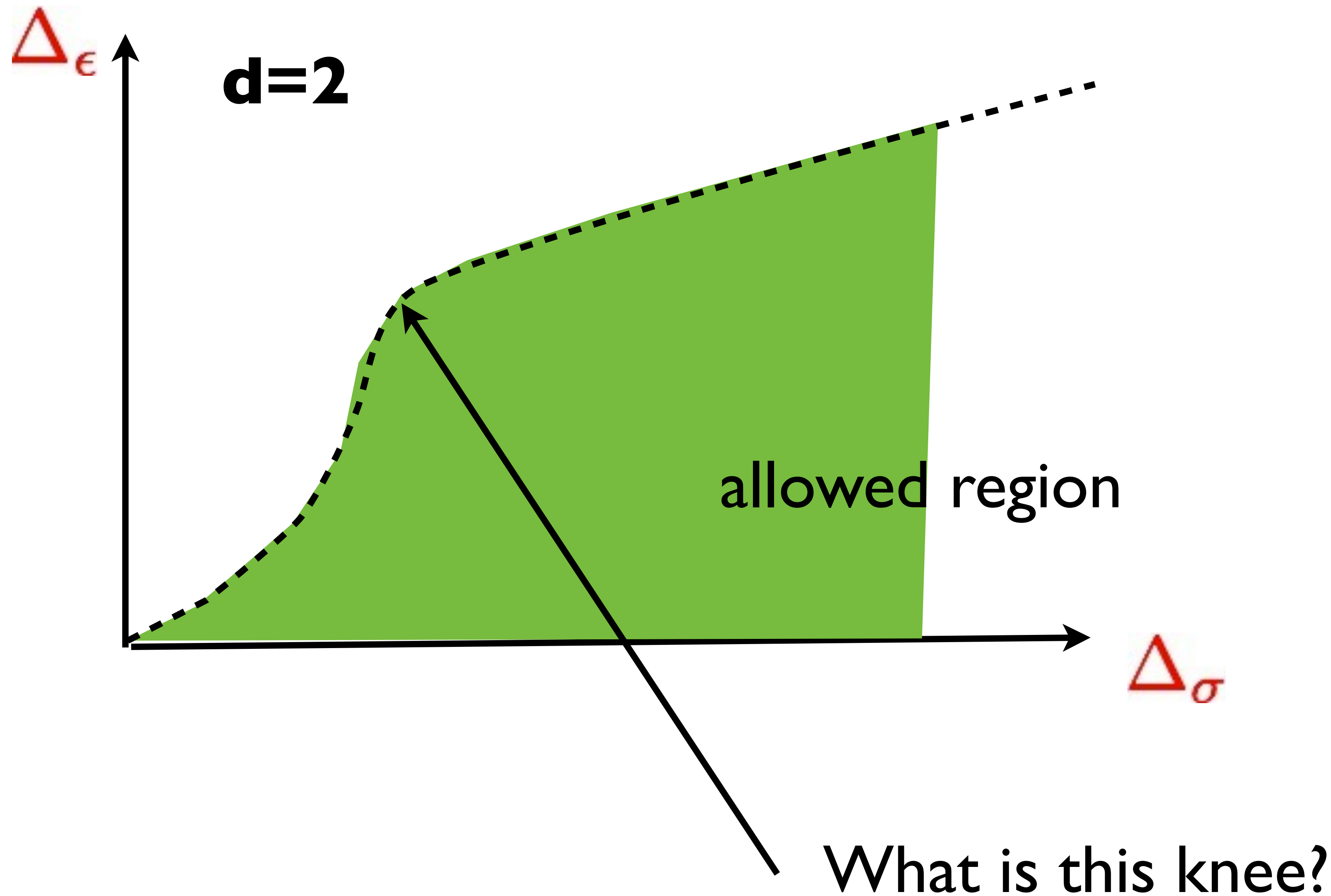


It's the 2D Ising model!
$\Delta_\sigma = 1/8$, $\Delta_\varepsilon = 1$

Extracting d=3 Ising critical exponents

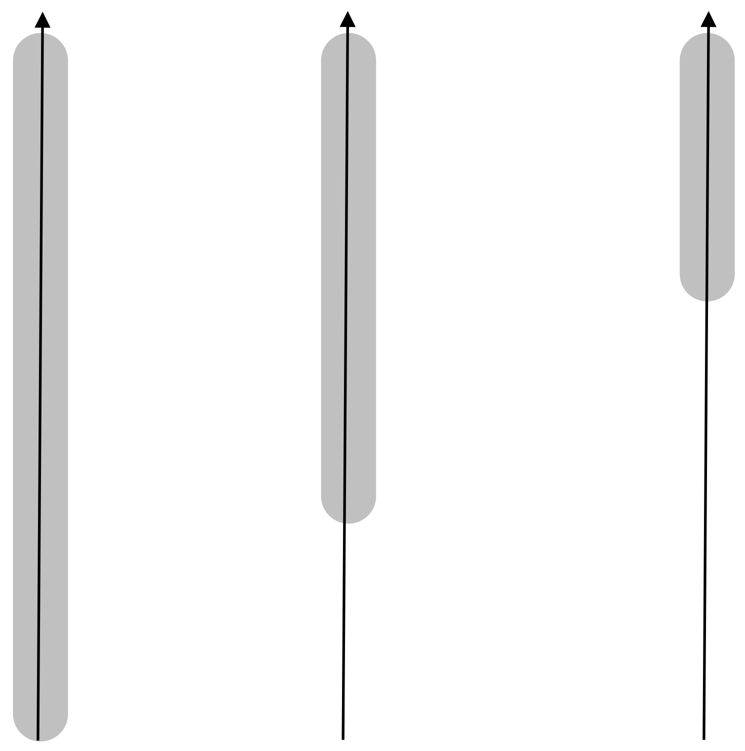
Idea 0:

Look for the knee on the boundary of allowed region in $(\Delta_\sigma, \Delta_\varepsilon)$ plane

Origin of the knee

Origin of the knee

Allowed OPE spectrum:



$\ell = 0$ $\ell = 2$ $\ell = 4$

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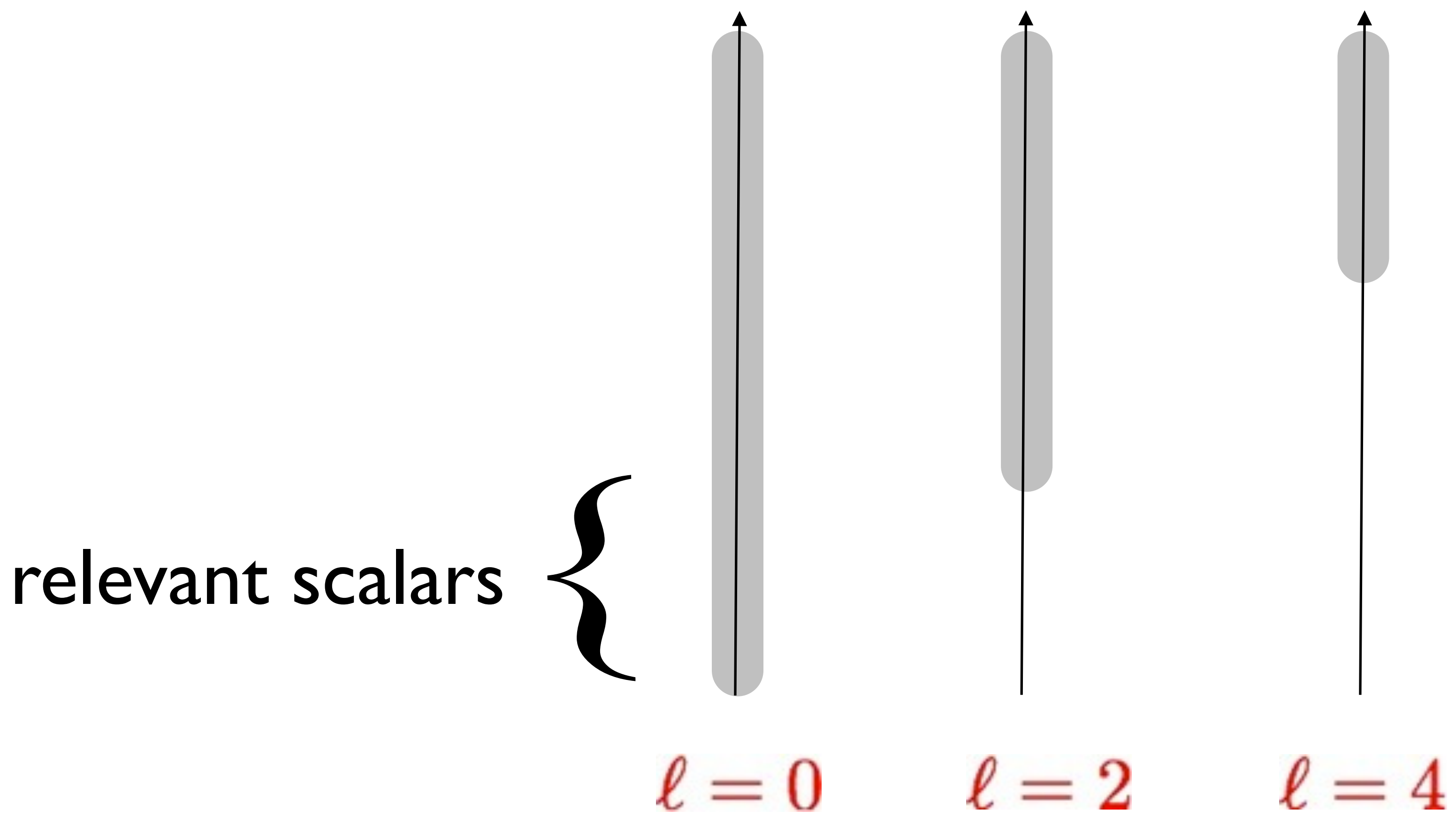

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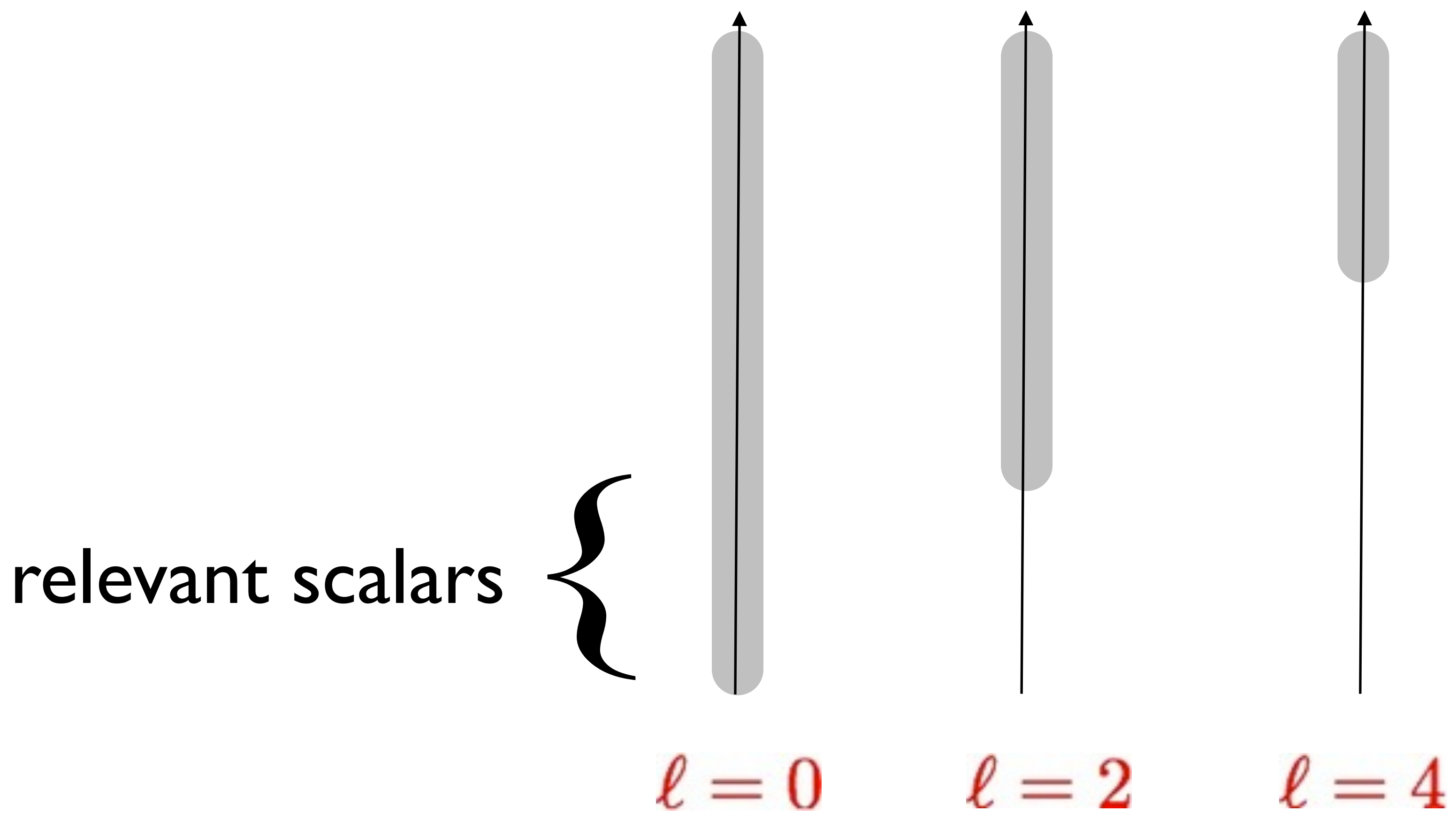

For Δ_σ <1/8 solutions to crossing symmetry have always ≥ 2 relevant scalars
(while Ising model has only one)

Impose condition that no other scalars in OPE (apart from ε) with $\Delta<2.5$

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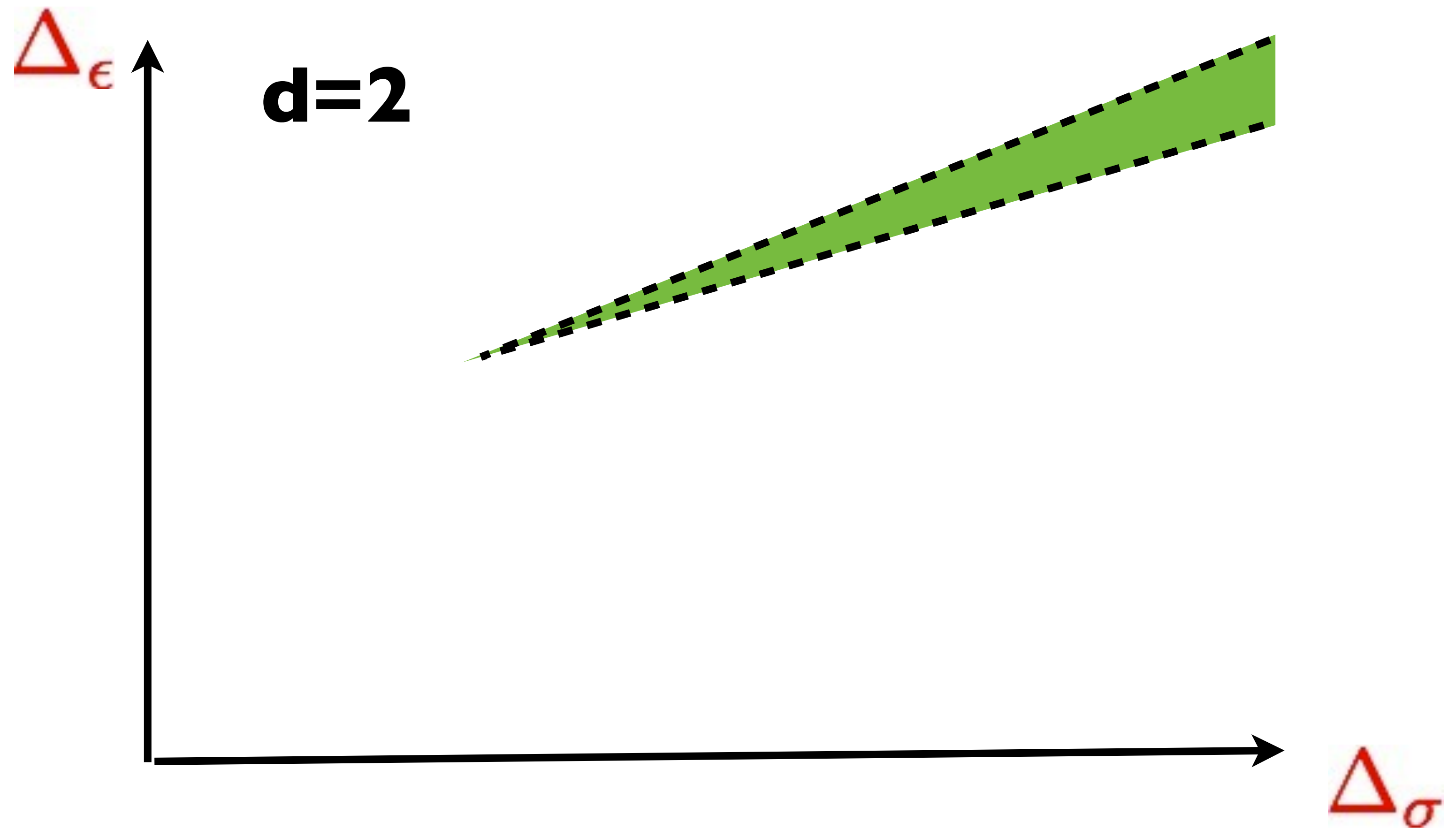

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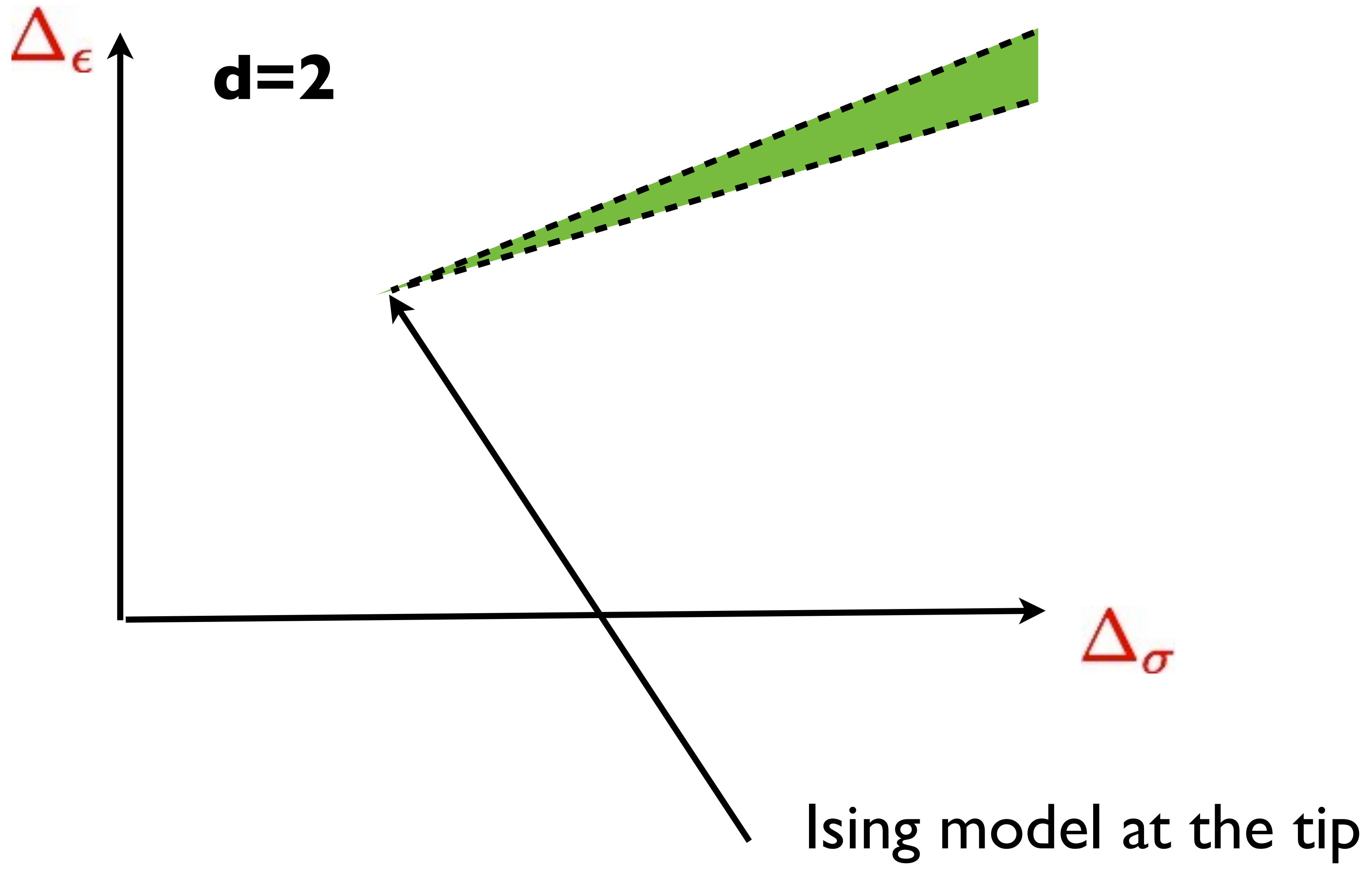

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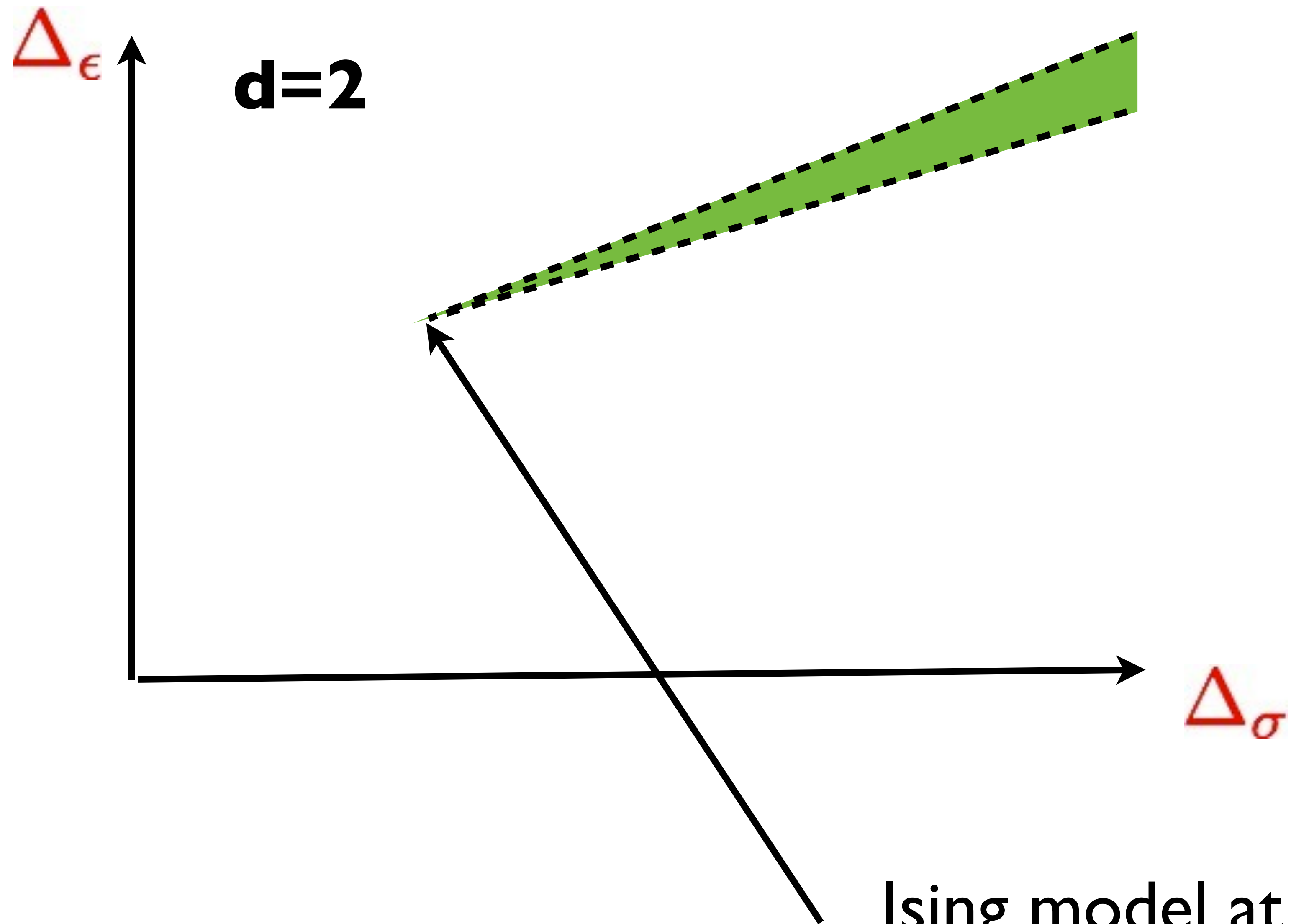

Allows much sharper determination of critical exponents

Why not yet applied in d=3?

Explicit conformal partial waves: Dolan,Osborn 2001

d=4
$$g_{\mathcal{O}}(u,v) = \frac{z\bar{z}}{z-\bar{z}}\left[k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right]$$

d=2
$$g_{\mathcal{O}}(u,v) = k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})$$

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z})$$

$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},\beta;x\right)$$

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In **d=3** equally simple expressions are not yet known
There exist double power series in (u, 1-v) which can be used but with more difficulty