Dear Colleagues,

Paolo Lodone is about to graduate from the Scuola Normale of Pisa. His expertise is Beyond the Standard Model phenomenology. He has worked on composite Higgs, low-energy supersymmetry, and TeV scale gravity. I know him quite well since I was an assistant professor at the Scuola Normale during his Master's thesis and the beginning of his PhD, and because we have a history of collaboration (see below). I think he is a very talented student, with strong technical skills. He is not afraid of hard problems which involve conceptual issues and which do not have ready-made solutions. Actually, I believe he prefers this kind of problems. I would classify him a "thinker", i.e. someone who may not be incredibly fast at the onset of a project, but is eventually able to see farther, endure longer, and make nontrivial contribution rather than just follow advisor's directions. He's the strongest among several phenomenology students who are finishing in Pisa this year, including as well the students of the previous years going back perhaps as far as the stellar generation of Contino, Papucci etc.

Here's how I got to know Paolo. For his first PhD project I proposed him the problem of QCD radiation in trans-Planckian scattering. As explained in a 2001 paper of Giudice, Rattazzi and Wells, small-angle trans-Planckian scattering is a complementary signal of TeV-scale gravity scenarios. It is not as widely acclaimed as the black hole production, but it has an advantage of being under theoretical control. However, QCD effects were never properly included in the calculation of the scattering amplitude. I knew about this problem from Riccardo Rattazzi, and I was also interested in it as a warmup for the more difficult problem of gravitational radiation. I had an idea how this could be approached so I suggested that Paolo look into this. Now, this was not the simplest thing to start your PhD with, but it proved an impressive test of Paolo's abilities. (Paolo's Master's thesis - his first paper - was on the electroweak precision tests in the composite Higgs boson scenario. This was an interesting contribution to the subject and is recognized as such in the later literature. But he had no prior experience with Extra-dimensional theories.)
Many reasons to think about CFT’s
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UV structure Beyond the Standard Model
- ad hoc CFT hidden sectors (unparticles)
- EWSB sector (Walking and Conformal Technicolor)
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- Quantum gravity in AdS
- Inflation (dS)
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Critical phenomena
- statistical mechanics
- quantum condensed matter (=quantum criticality)
Classical $O(N)$-invariant ferromagnet in $d=3$

$$Z = \exp \left[ -\frac{1}{T} \sum_{\langle ij \rangle} \sigma_i \cdot \sigma_j \right]$$

$\sigma_i \in S^{N-1}$
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$N=1$: Ising model \( (\sigma = \pm 1) \)
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Near critical temperature:

$$\langle \sigma(r)\sigma(0) \rangle \sim \frac{1}{r^{2\Delta_\sigma}} e^{-r/\xi(T)}$$
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focus on this

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\[ \xi(T) \to \infty \quad (T \to T_c) \]

correlation length
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$\xi(T) \to \infty \quad (T \to T_c)$

At $T= T_c$ theory is 1) scale invariant
2) conformal
Special Conformal transformation

\[ \delta_\kappa x_a = 2(\kappa \cdot x)x_a - x^2 \kappa_a \]

(preserves orthogonality of coordinate grid; locally looks like dilation)
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(preserves orthogonality of coordinate grid; locally looks like dilation)

Are there interesting scale invariant theories without full conformal invariance?

Critical Exponents (universal)
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2) Correlation length exponent \( \nu \):
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Related to the dimension of another local field, \( \varepsilon(x) \):

\[ \nu = \frac{1}{d - \Delta_\varepsilon} \]
Energy density field, $\varepsilon(x)$

$\varepsilon \sim \sigma^2$
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Lagrangian describing the near-critical system:

$$\mathcal{L}_{CFT} + \frac{T/T_c - 1}{a^{d-\Delta_\varepsilon}} \varepsilon(x)$$
Energy density field, $\varepsilon(x)$

$\varepsilon \sim \sigma^2$

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atomic spacing
Energy density field, $\varepsilon(x)$

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Lagrangian describing the near-critical system:

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$$\mathcal{L}_{CFT} + \frac{T/T_c - 1}{a^{d-\Delta\varepsilon}} \varepsilon(x)$$

$$\sim \frac{1}{\xi(T)^{d-\Delta\varepsilon}}$$

Correlation length develops at the length scale where perturbation grows to $O(1)$ in strength

$$\nu = \frac{1}{d - \Delta\varepsilon}$$
Flows

In particular from $\nu > 0$ it follows $\Delta_\varepsilon < d$ (relevant operator)
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Important: $\varepsilon$ is the only relevant operator which is singlet under $Z_2$ symmetry $\sigma \rightarrow -\sigma$
(otherwise multicriticality)
How to determine critical exponents

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\[ \Delta_\varepsilon = 1.412(1) \]
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- Laboratory measurements
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- Lattice Monte-Carlo simulations
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- Laboratory measurements

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- Field theory techniques in 4-\(\epsilon\) dimensions
Study scalar field theory in 4-$\epsilon$ dimensions

$$\mathcal{L} = (\partial \phi)^2 + \lambda \phi^4$$

$$\beta_\lambda = -\epsilon \lambda + \frac{\lambda^2}{16\pi^2} + \ldots \rightarrow 0$$

$$\frac{\lambda_*}{16\pi^2} = O(\epsilon) \ll 1 \quad \text{weakly coupled fixed point}$$
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$$\frac{\lambda_*}{16\pi^2} = O(\epsilon) \ll 1$$

weakly coupled fixed point

Compute critical exponents at $\epsilon \ll 1$ and then extrapolate to $\epsilon = 1$

Works pretty well but not to arbitrary accuracy (divergent series)
I think the epsilon expansion ended the subject in the practical sense.

You can calculate more or less what you want with good accuracy but aesthetically the subject is not closed yet.

It's possible that there will be classification of fixed points in three dimensions. But that's just dreams.

A.M. Polyakov, 2003 interview
Conformal bootstrap

Consistency eq. four 4pt function

\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \]
Conformal bootstrap

Consistency eq. four 4pt function

\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \]

OPE:

\[ \sigma(x_1) \sigma(x_2) = 1 + \epsilon + \ldots \]
Conformal bootstrap

Ferrara, Gatto, Grillo 1973
Polyakov 1974

Consistency eq. four 4pt function

\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \]

OPE:

\[ \langle \sigma \sigma \rangle \neq 0 \]

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\[ \langle \sigma \sigma \epsilon \rangle \neq 0 \]

Conformal symmetry fixes:

\[ \langle \sigma(x) \sigma(y) \epsilon(0) \rangle = \frac{\lambda}{|x-y|^{2\Delta_{\sigma} - \Delta_{\epsilon}} |x|^{2\Delta_{\epsilon}} |y|^{2\Delta_{\epsilon}}} \]
Conformal OPE

$$\sigma(x_1) \sigma(x_2) = \sum_O \lambda_O C(x_1 - x_2, \partial x_2) O(x_2)$$
Conformal OPE

\[ \sigma(x_1) \sigma(x_2) = \sum_O \lambda_O \, C(x_1 - x_2, \partial_{x_2}) \, O(x_2) \]

fixed by conformal symmetry
Conformal OPE

\[ \sigma(x_1) \sigma(x_2) = \sum_{O} \lambda_O C(x_1 - x_2, \partial x_2) O(x_2) \]

fixed by conformal symmetry

\[ O = O_{\Delta}^{(l)} \]

\[ l = 2, 4, 6, \ldots \]

\[ \Delta \geq l + d + 2 \quad (\geq d/2 - 1, l = 0) \]
\left< \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \right>
\[
\langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle
\]

Conformal partial wave

\[
= \sum_{O} \lambda_{O}^{2}
\]
\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \]

Conformal partial wave

\[ \langle \sigma \sigma \sigma \sigma \rangle = \frac{g(u, v)}{r_{12}^{2\Delta_{\sigma}} r_{34}^{2\Delta_{\sigma}}} \]

\[ u = \left( \frac{r_{12} r_{34}}{r_{13} r_{24}} \right)^2 \quad v = \left( \frac{r_{14} r_{23}}{r_{13} r_{24}} \right)^2 \]
\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \]

\[
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Conformal partial wave

\[ \langle \sigma \sigma \sigma \sigma \rangle = \frac{g(u, v)}{r_{12}^{2\Delta_{\sigma}} r_{34}^{2\Delta_{\sigma}}} \]

\[
u = \left( \frac{r_{12} r_{34}}{r_{13} r_{24}} \right)^{2} \]

\[ g(u, v) = 1 + \sum_{O} \lambda_{O}^{2} g_{O}(u, v) \]

known functions of \( u, v \)
Crossing symmetry/OPE associativity

\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \overset{\text{OPE}}{=} \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \]
Crossing symmetry/OPE associativity

\[ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \ = \ \langle \sigma(x_1) \sigma(x_2) \sigma(x_3) \sigma(x_4) \rangle \]

\[ g(u, v) = \left( \frac{u}{v} \right)^{\Delta_{\sigma}} g(v, u) \]

\[ g(u, v) = 1 + \sum_{O} \lambda_{O}^{2} g_{O}(u, v) \]
Crossing symmetry/OPE associativity

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\[ g(u, v) = (\frac{u}{v})^{\Delta_\sigma} g(v, u) \]

\[ g(u, v) = 1 + \sum_O \lambda^2_O g_O(u, v) \]

Conformal bootstrap equation for CFT couplings and spectrum
(no progress for 30 years)
Resurrecting Conformal Bootstrap

Rattazzi, S.R., Tonni, Vichi 2008
S.R., Vichi 2009
Caracciolo, S.R. 2009
Rattazzi, S.R., Vichi 2010
Vichi 2011

\[
\Delta_\epsilon \quad \Delta_\sigma
\]

allowed region

free scalar
Resurrecting Conformal Bootstrap

Motivated by Conformal Technicolor

\[ \Delta_\epsilon \]

\[ \Delta_\sigma \]

\[ d=4 \]

allowed region

free scalar

Rattazzi, S.R., Tonni, Vichi 2008
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Caracciolo, S.R. 2009
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Vichi 2011
Exercise in $d=2$

S.R., Vichi 2009

$\Delta_\epsilon$

$\Delta_\sigma$

$d=2$

allowed region
Exercise in $d=2$

What is this knee?
Exercise in $d=2$

What is this knee?

It’s the 2D Ising model!

$\Delta_\sigma = 1/8, \Delta_\varepsilon = 1$
Extracting d=3 Ising critical exponents

Idea 0:
Look for the knee on the boundary of allowed region in \((\Delta_\sigma, \Delta_\epsilon)\) plane
Origin of the knee
Origin of the knee

Allowed OPE spectrum:

$\ell = 0 \quad \ell = 2 \quad \ell = 4$
Origin of the knee

Allowed OPE spectrum:

relevant scalars \{ \ell = 0 , \ell = 2 , \ell = 4 \}
Origin of the knee

Allowed OPE spectrum:

\[ \text{relevance scalars} \{ \]

\[ \ell = 0 \quad \ell = 2 \quad \ell = 4 \]

For \( \Delta_\sigma <1/8 \) solutions to crossing symmetry have always \( \geq 2 \) relevant scalars
(while Ising model has only one)
Impose condition that no other scalars in OPE (apart from $\varepsilon$) with $\Delta < 2.5$
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Ising model at the tip
Impose condition that no other scalars in OPE (apart from $\varepsilon$) with $\Delta < 2.5$.

This condition allows much sharper determination of critical exponents in the Ising model at the tip ($d=2$).
Why not yet applied in $d=3$?

Explicit conformal partial waves: Dolan, Osborn 2001

$d=4$ \[ g_O(u, v) = \frac{z\bar{z}}{z - \bar{z}} \left[ k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \right] \]

$d=2$ \[ g_O(u, v) = k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z}) \]

\[ u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z}) \]

\[ k_{\beta}(x) \equiv x^{\beta/2} \, _2F_1 \left( \frac{\beta}{2}, \frac{\beta}{2}, \beta; x \right) \]
Why not yet applied in \( d=3 \)?

Explicit conformal partial waves: Dolan, Osborn 2001

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g_O(u, v) = \left( \frac{z \bar{z}}{z - \bar{z}} \right) \left[ k_{\Delta+i}(z) k_{\Delta-i-2}(\bar{z}) - (z \leftrightarrow \bar{z}) \right]
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\[
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\]

\[
u = z \bar{z}, \quad v = (1 - z)(1 - \bar{z})
\]

\[
k_\beta(x) \equiv x^{\beta/2} \binom{\beta}{2} \binom{\beta}{2} \binom{\beta}{\beta} ; x
\]

In \( d=3 \) equally simple expressions are not yet known.

There exist double power series in \((u, 1-v)\) which can be used but with more difficulty.