Scalar potential of the general N-Higgs-doublet model

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Outline

N-Higgs-doublet model

- 2HDM as a simple (yet rich) bSM extension of the Higgs mechanism;
- Pursuing the idea of Higgs doublet generations further — *N*-Higgs-doublet model.
- Many specific variants of NHDM for $N \geq 3$ were suggested (Weinberg 3HDM, Adler 4HDM, SUSY version of ν2HDM, private Higgs, etc).
- The focus has been on the fermion mass matrices. However, there is rich physics in the scalar sector barely touched.
- Very little is known about what in principle can happen in the scalar sector of NHDM.

Understanding the scalar sector of the general NHDM can become a guide to novel phenomenological possibilities.

NHDM vs. 2HDM

Examples of what is known in general 2HDM (at the tree-level):

- At most two local minima of the Higgs potential.
- Non-coexistence of neutral and charge-breaking minima; of *CP*-conserving and spontaneously *CP*-violating minima.
- 6 classes of symmetries possible.
- Explicit reparametrization-invariant conditions for existence and spontaneous breaking of these symmetries.
- Phase diagram of the model, phase transitions and critical exponents.

We want a method powerful enough to give similar results for a general *N*-Higgs-doublet model.

Describing NHDM

We have N doublets of complex Higgs fields ($4N$ degrees of freedom):

$$\phi_a = \begin{pmatrix} \phi_a^+ \\ \phi_a^0 \end{pmatrix}, \quad a = 1, \ldots, N.$$

The Higgs potential depends on them via $(\phi_a^\dagger \phi_b)$:

$$V = Y_{\bar{a}b}(\phi_a^\dagger \phi_b) + Z_{\bar{a}b\bar{c}d}(\phi_a^\dagger \phi_b)(\phi_c^\dagger \phi_d)\,,$$

with N^2 independent components in Y and $N^2(N^2+1)/2$ independent components in Z.
$N = 2$: there are 4 Y's and 10 Z's;
$N = 3$: there are 9 Y's and 45 Z's, etc.

The v.e.v.'s $\langle\phi_a\rangle$ come from the global minimum of the potential.

Electroweak vacuum: $\langle\phi_a\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\,, \quad a = 1,\dots,N\,.$

Neutral vacuum parametrized with $2N-1$ parameters:

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}\,, \quad \langle\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_a e^{i\xi_a} \end{pmatrix}\,, \quad a \geq 2\,,$$

Charge-breaking vacuum parametrized by $4N-4$ parameters:

$$\langle\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_1 \end{pmatrix}\,, \quad \langle\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} u_2 \\ v_2 e^{i\xi_2} \end{pmatrix}\,, \quad \langle\phi_a\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} u_a e^{i\zeta_a} \\ v_a e^{i\xi_a} \end{pmatrix}\,, \quad a \geq 3\,,$$

The main problem in NHDM: the general potential cannot be minimized explicitly (coupled algebraic equations of high order) $\to$ other methods are needed to learn at least something about the general NHDM.

For 2HDM several basis-invariant techniques were developed:

- tensorial formalism [*Botella, Silva (1995); Haber, Gunion, Davidson, O'Neil (2005-2006)*]
- geometric approach [*Maniatis, Nachtmann, Nagel von Mannteuffel; Ivanov; Nishi; (2004-2008)*].

Many non-trivial results (number and coexistence of minima, symmetries and their violation, phase diagram) were obtained without explicit minimization of the potential.

Generalization of this approach to N-doublets leads to more non-trivial mathematics.

K-matrix

Following [*Nagel (2004), Maniatis et al (2006)*] we introduce the hermitian positive-semidefinite $N \times N$ matrix $K_{ab} \equiv (\phi_b^\dagger \phi_a)$ with rank ≤ 2. Expressing it via the $SU(N)$ generators λ_i gives:

$$K_{ab} = r_0 \cdot \sqrt{\frac{2}{N(N-1)}}\mathbf{1}_N + r_i \lambda_i\,, \quad i = 1, \dots N^2 - 1\,.$$

These vectors $r^\mu = (r_0, r_i) \in \mathbb{R}^{N^2}$ are in one-to-one correspondence with gauge orbits in the $4N$-dimensional space of Higgs fields; they form the orbit space of NHDM.

Orbit space

In 2HDM the orbit space was just the surface and the interior of the 4D-cone:

$$r_0 > 0\,, \quad r_0^2 - r_i^2 \geq 0\,.$$

In NHDM with $N > 2$, the orbit space is a complicated shape immersed inside an N^2-dimensional cone.

In [*Ivanov, Nishi (2010)*] it was fully characterized

- algebraically via additional basis-invariant algebraic equations on r_i,
- geometrically in terms of complex projective plane $\mathbb{CP}^{N-1}$.

The tolopogy of NHDM orbit space is much more complicated than in 2HDM $\to$ possibility of novel (metastable) vacuum configurations in NHDM.

Links to quantum information theory

- In quantum information theory, one studies evolution of (coupled) quantum N-level states: qubits ($N = 2$) or qudits (general N).
- Mathematically, describing the space of states for a single qudit is very similar to construction of the orbit space in NHDM.

information theory	NHDM
Bloch sphere	orbit space
coherence vector	$\vec{n} \equiv \vec{r}/r_0$
density matrix	K-matrix
pure state	neutral vacuum
(rank-2) mixed state	charge-breaking vacuum

The NHDM potential

- Switching from the space of doublets ϕ_a to the $r^\mu \equiv (r_0, r_i)$ space complicates the description of the orbit space.
- But there is a pay-off: the Higgs potential is drastically simplified:

$$V = -M_\mu r^\mu + \frac{1}{2}\Lambda_{\mu\nu} r^\mu r^\nu \,.$$

  All free parameters in the potential are nicely grouped into two basic geometric objects: M_μ and $\Lambda_{\mu\nu}$.
- All the physically relevant information is encoded in the relative orientation between M_μ, the eigenaxes of $\Lambda_{\mu\nu}$, and the orbit space.

Minimizing NHDM potential

Although the explicit algebraic minimization of the potential is impossible, a geometric technique based on equipotential surfaces was developed that allows for study of some properties of the minima [*Ivanov (2010)*].

The idea is that the minima correspond to points where two algebraic surfaces (an equipotential surface and the orbit space) with well-defined algebraic degrees touch each other.

The example of 3HDM

The space of Higgs fields ϕ_a is 12D. The Higgs potential depends on 9 bilinears $(\phi_a^\dagger \phi_b)$.

$$K_{ab} = r_0 \cdot \frac{1}{\sqrt{3}} \mathbf{1}_3 + r_i \lambda_i \,.$$

where λ_i, $i = 1, \ldots, 8$, are Gell-Mann matrices.
The orbit space is an 8D-manifold embedded in the 9D space of (r_0, r_i) defined by

$$r_0 \geq 0\,, \quad \vec{n}^2 \leq 1\,, \quad \sqrt{3} d_{ijk} n_i n_j n_k = \frac{3\vec{n}^2 - 1}{2}\,.$$

where $\vec{n} \equiv \vec{r}/r_0$.

The example of 3HDM

- Charge-breaking manifold corresponds to $\vec{n}^2 < 1$. Its dimension in the $\vec{n}$-space is 7, algebraic degree is 3.
- Neutral manifold corresponds to $\vec{n}^2 = 1$. Its dimension is 4, algebraic degree is 6.

Consequences:

- the potential can have at most three charge-breaking or six neutral minima.
- any finite symmetry group of 3HDM with order > 6, if it is spontaneously broken, cannot break completely.

Some other results for 3HDM

- Degeneracy of minima does not necessarily imply a symmetry of the potential.
- Minima of different symmetries can coexist and can be degenerate.
- Charge-breaking/restoring phase transitions and transitions from *CP*-conserving to spontaneously *CP*-violating transitions can be of the first order even at the tree level. This can become interesting in the context of thermal evolution of the early Universe.

Symmetries

A big motivation to study several Higgs doublets is that symmetries can be encoded in the Higgs potential and then transmitted to the fermionic sector.

- Which groups can be realized as symmetry groups of the NHDM Higgs potential?
- How can these groups spontaneously break after EWSB?
- How to find the symmetry group of a given potential?

In 2HDM, all these questions have been answered. In NHDM for $N > 2$ no answer is known in the general case. Only few very specific symmetry groups have been constructed and studied.

Realizable symmetry groups

Definition:
we call a symmetry group G realizable if there exists a G-symmetric potential which is not symmetric under a larger symmetry group containing G.

Example:
$(\phi_1^\dagger \phi_1)$ is symmetric under cyclic group $\mathbb{Z}_p$, for any integer p, generated by phase rotations by $2\pi/p$. But each $\mathbb{Z}_p$ is not a realizable symmetry because $(\phi_1^\dagger \phi_1)$ is $U(1)$-symmetric under arbitrary phase rotation.

Only realizable groups show the true symmetries of potentials.

Abelian symmetries

In [*Ebrahimi-Keus, Ivanov, Vdovin, in progress*] we developed a strategy that yields all realizable abelian symmetry groups for a given N.

- For the strategy and the results see the talk by *V. Ebrahimi-Keus*.
- In particular for 3HDM, the realizable finite abelian symmetry groups are

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2\times\mathbb{Z}_2, \quad \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2.$$

  Note that order of any of these groups is divisible only by two primes: 2 and 3.

Non-abelian finite groups for 3HDM

Knowing all abelian finite groups can help reconstruct realizable non-abelian finite groups G.

This works especially well for 3HDM.

- Order of G must be $2^a 3^b$, no other prime divisor allowed.
- Then the Burnside's $p^a q^b$ theorem states that G must be solvable.
- A solvable group can be reconstructed by successive split extensions.

Preliminary results for unitary groups: any subgroup of the cubic symmetry group, and the $\mathbb{Z}_3 \rtimes S_3$.

Frustrated symmetries

In 2HDM any symmety imposed on the potential, upon an appropriate choice of the coefficients, could be conserved (or spontaneously broken) in the vacuum state after EWSB.

In NHDM for $N \geq 3$ a novel class of symmetries exists: symmetries which are present in the potential but necessarily broken in the vacuum state [*Ebrahimi-Keus, Ivanov (2011)*].

We termed them frustrated symmetries, because the origin of this phenomenon is mathematically similar to frustration in condensed matter physics.

Conclusions

- Understanding properties of the general NHDM is needed to guide phenomenological models, but it requires rather complicated mathematical tools.
- We have developed such tools, both algebraic and geometric, and used them to study several aspects of the scalar sector of NHDM (number and coexistence of minima, symmetries and their breaking).
- We developed a strategy that finds all realizable abelian symmetry groups for any number of doublets.
- We are on the way to finding all realizable discrete symmetry groups for 3HDM.

Just an example: the Higgs potential in general 2HDM is $V = V_2 + V_4$:

$$V_2 = -\frac{1}{2}\left[m_{11}^2(\phi_1^\dagger\phi_1) + m_{22}^2(\phi_2^\dagger\phi_2) + m_{12}^2(\phi_1^\dagger\phi_2) + m_{12}^{2\,*}(\phi_2^\dagger\phi_1)\right] ;$$

$$V_4 = \frac{\lambda_1}{2}(\phi_1^\dagger\phi_1)^2 + \frac{\lambda_2}{2}(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1)$$
$$+\frac{1}{2}\lambda_5(\phi_1^\dagger\phi_2)^2 + \left\{\left[\lambda_6(\phi_1^\dagger\phi_1) + \lambda_7(\phi_2^\dagger\phi_2)\right](\phi_1^\dagger\phi_2) + \mathrm{h.c.}\right\}$$

It contains 4 free parameters m_{ab}^2 and 10 λ's.

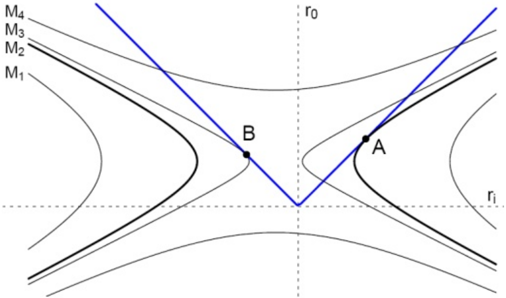
Geometric look at minimization of the Higgs potential

Consider $V(r) = -M_\mu r^\mu + \Lambda_{\mu\nu} r^\mu r^\nu/2$ not just in the physical manifold, but in the entire r^μ space.

Choose some real number C. An equipotential surface M_C: all r^μ that give the potential $V(r) = C$.

- Each equipotential surface is an (N^2-1)-dimensional quadric (hyperboloid, ellipsoid, etc.) embedded in the N^2-dimensional space r^μ.
- M_C and $M_{C'}$ do not intersect.
- The family of nested equipotential surfaces covers the entire space r^μ.

Minimization of the potential: the unique equipotential surface that barely touches physical manifold gives the global minimum.



M_1 does not intersect LC^+. M_2 touches LC^+ at a single point A (global minimum). M_3 touches LC^+ at point B (local minimum).

Despite the condition involving d_{ijk} is rather complicated, the shape of the physical manifold can be explained in a rather compact way.

- In the $\vec{n}$-space, the neutral manifold is $\mathbb{CP}^2$ (two-complex-dimensional projective space).
- The charge-breaking manifold is a "self-join" of the neutral manifold: if points $\vec{n}$ and $\vec{n}'$ belong to the neutral manifold, then the line segment $c\vec{n} + (1-c)\vec{n}'$, $c = (0,1)$, belongs to the charge-breaking manifold.

This allows us to write down the symmetries of the physical manifold and, to a certain degree, visualize the manifold.

Reparametrization transformations

Reparametrization transformations mixing different doublets ϕ_a must be

- either unitary U acting as $\phi_a \to U_{ab}\phi_b$,
- or antiunitary $U_{CP} = U \cdot CP$: $\phi_a \to U_{ab}\phi_b^*$.

Unitary transformations (up to overall phase rotation) form the group

$$PSU(N) \simeq SU(N)/\mathbb{Z}_N\,.$$

To include anti-unitary transformations, note that $CP \cdot U \cdot CP = U^*$, which defines the action of CP on the group of unitary transformations. Therefore, the reparametrization group G is a semidirect product

$$G = PSU(N) \rtimes \mathbb{Z}_2\,.$$

Any symmetry group of the potential must be a subgroup of this group.