Scalars from modified gravity

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Why new scalar degree of freedom in gravity and cosmology?

The standard FLRW cosmology with small superimposed perturbations has a non-fundamental scalar degree of freedom - scalar (density) perturbations in the matter (CDM and baryons) and radiation components. However, the primordial power spectrum of one half of these perturbations (the growing mode) may be arbitrary, while the second half (the decaying mode) is absent at all (that, in particular, is the reason for the existence of observed acoustic (Sakharov) oscillations in the matter and radiation power spectra).

To obtain the primordial spectrum of the growing mode from an in-vacuum state using the effect of particle-antiparticle creation by gravitational fields, some more fundamental scalar degree of freedom is needed. This refers to both inflation and its alternatives including bouncing scenarios, the Pre-Big-Bang scenario, the ekpyrotic scenario, etc.

The whole known part of the history of our Universe in one line, according to the standard cosmological scenario:

$$? \longrightarrow DS \Longrightarrow FLRWRD \Longrightarrow FLRWMD \Longrightarrow \overline{DS} \longrightarrow ?$$

It contains two quasi-DS stages for which dark energy (DE) is needed:

1) inflation in the early Universe – primordial DE,

2) present accelerated expansion of the Universe – present DE.

Remarkable qualitative similarity of DS and $\overline{DS}$ makes possible (though not necessary) combined description of both DS stages (both types of DE) using one class of models.

Possible forms of DE

- Physical DE
  New non-gravitational field of matter. DE proper place – in the rhs of gravity equations.
- Geometrical DE
  Modified gravity. DE proper place – in the lhs of gravity equations.
- Λ - intermediate case.

Generically, DE can be both physical and geometrical, e.g. in the case of a non-minimally coupled scalar field or, more generically, in scalar-tensor gravity. So, there is no alternative "(either) dark energy or modified gravity".

$f(R)$ gravity

The simplest model of modified gravity (= geometrical dark energy) considered as a phenomenological macroscopic theory in the fully non-linear regime and non-perturbative regime.

$$S = \frac{1}{16\pi G} \int f(R)\sqrt{-g}\, d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R^{\mu}_{\mu} \ .$$

Field equations

$$\frac{1}{8\pi G}\left(R_\mu^\nu - \frac{1}{2}\,\delta_\mu^\nu R\right) = -\left(T^\nu_{\mu\,(vis)} + T^\nu_{\mu\,(DM)} + T^\nu_{\mu\,(DE)}\right)\ ,$$

where $G = G_0 = const$ is the Newton gravitational constant measured in laboratory and the effective energy-momentum tensor of DE is

$$8\pi G T^\nu_{\mu\,(DE)} = F'(R)\,R_\mu^\nu - \frac{1}{2}\,F(R)\delta_\mu^\nu + \left(\nabla_\mu\nabla^\nu - \delta_\mu^\nu\nabla_\gamma\nabla^\gamma\right)F'(R)\,.$$

Because of the need to describe DE, de Sitter solutions in the absence of matter are of special interest. They are given by the roots $R = R_{dS}$ of the algebraic equation

$$Rf'(R) = 2f(R)\ .$$

Degrees of freedom

I. In quantum language: particle content.
1. Graviton – spin 2, massless, transverse traceless.
2. Scalaron – spin 0, massive, mass - R-dependent:
$m_s^2(R) = \frac{1}{3f''(R)}$ in the WKB-regime.

II. Equivalently, in classical language: number of free functions of spatial coordinates at an initial Cauchy hypersurface.
Six, instead of four for GR – two additional functions describe massive scalar waves.

Thus, $f(R)$ gravity is a non-perturbative generalization of GR. It is equivalent to scalar-tensor gravity with $\omega_{BD} = 0$ (if $f''(R) \neq 0$).

Why R-dependence only?

For almost all other geometric invariants – $R_{\mu\nu}R^{\mu\nu}$, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, $R_{,\mu}R^{,\mu}$ etc. (where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor) – ghosts appear if the theory is taken in full, in the non-perturbative regime.

The only known exception: $f(R,G)$ with $f_{RR}f_{GG} - f_{RG}^2 = 0$, where $G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the Gauss-Bonnet invariant, does not possess ghosts but has other problems.

For $f_{RR}f_{GG} - f_{RG}^2 \neq 0$, a ghost was found very recently (A. De Felice and T. Tanaka, Progr. Theor. Phys. **124**, 503 (2010)).

Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\,, \qquad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

Conditions for viable $f(R)$ models

I. Conditions of classical and quantum stability:

$$f'(R) > 0 \; , \quad f''(R) > 0 \; .$$

Even the saturation of these inequalities should be avoided:
1. $f'(R_0) = 0$: a generic anisotropic space-like curvature singularity forms.
2. $f''(R_0) = 0$: a weak singularity forms, loss of predictability of the Cauchy evolution.

$$a(t) = a_0 + a_1(t - t_s) + a_2(t - t_s)^2 + a_3|t - t_s|^{5/2} + \dots \; .$$

The metric in C^2, but not C^3, continuous across this singularity, and there is no unambiguous relation between the coefficients a_3 for $t < t_s$ and $t > t_s$. Also, the equivalence of $f(R)$ gravity to scalar-tensor gravity with $\omega_{BD} = 0$ is broken in its vicinity.

II. Conditions for the existence of the Newtonian limit:

$$|F| \ll R, \quad |F'| \ll 1, \quad RF'' \ll 1$$

for $R \gg R_{now}$ and up to some very large R.
The same conditions for smallness of deviations from GR.
III. Laboratory and Solar system tests.
No deviation from the Newton law up to 50 μ.
No deviation from the Einstein values of the post-Newtonian coefficients β and γ up to 10^{-4} in the Solar system.
IV. Existence of a future stable (or at least metastable) de Sitter asymptote:

$$f'(R_{dS})/f''(R_{dS}) \geq R_{dS} \ .$$

Required since observed properties of DE are close to that of a cosmological constant.

V. Cosmological tests:
among them the anomalous growth of matter perturbations for recent redshifts

$$\left(\frac{\delta\rho}{\rho}\right)_m \propto t^{\frac{\sqrt{33}-1}{6}}$$

at the matter-dominated stage for $k \gg m_s(R)a$, where

$$m_s^2(R) = 1/3F''(R)\ .$$

Results in apparent discrepancy between the linear σ_8 and the primordial slope n_s estimated from CMB data (assuming GR) and from galaxy/cluster data separately.

VI. $f(R)$ cosmology should not destroy previous successes of present and early Universe cosmology in the scope of GR, including the existence of the matter-dominated stage driven by non-relativistic matter preceded by the radiation-dominated stage with the correct BBN and, finally, inflation.

Inflationary models in $f(R)$ gravity

1. The simplest one (Starobinsky, 1980):

$$f(R) = R + \frac{R^2}{6M^2}$$

with small one-loop quantum gravitational corrections producing the scalaron decay via the effect of particle-antiparticle creation by gravitational field (so all present matter is created in this way).

During inflation ($H \gg M$): $H = \frac{M^2}{6}(t_f - t), \quad |\dot{H}| \ll H^2$.

The only parameter M is fixed by observations – by the primordial amplitude of adiabatic (density) perturbations in the gravitationally clustered matter component:

$$M = 3.0 \times 10^{-6} M_{Pl} \, (50/N) \; ,$$

where $N \sim (50-55)$ is the number of e-folds between the first Hubble radius crossing during inflation of the present Hubble scale and the end of inflation, $M_{Pl} = \sqrt{G} \approx 10^{19}$ GeV.

Remains viable: $n_s = 1 - \frac{2}{N} \approx 0.96, \; r \equiv \frac{P_g}{P_\zeta} = \frac{12}{N^2} \approx 0.004$.

Observations: $n_s = 0.963 \pm 0.012; \; r < 0.24$ (95% CL).

The main and simplest alternative: the simplest scalar field inflationary model with $V(\phi) = \frac{m^2\phi^2}{2}$ and $m = M/\sqrt{2(1+r)} \approx 2.0 \times 10^{-6} M_{Pl}$ which produces the same n_s but the significantly larger $r = \frac{8}{N} \approx 0.15$.

2. Analogues of chaotic inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R} \, , \; |A''(R)| \ll \frac{A(R)}{R^2} \, .$$

3. Analogues of new inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1} \, , \; F''(R_1) \approx \frac{2F(R_1)}{R_1^2} \, .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

One viable microphysical model leading to such form of $f(R)$

A non-minimally coupled scalar field with a large negative coupling ξ (for this choice of signs, $\xi_{conf} = \frac{1}{6}$):

$$L = \frac{R}{16\pi G} - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \; .$$

Leads to $f' > 1$.
Recent development: the BEH inflation (F. Bezrukov and M. Shaposhnikov, 2008). In the limit $|\xi| \gg 1$, the BEH scalar tree level potential $V(\phi) = \frac{\lambda(\phi^2 - \phi_0^2)^2}{4}$ just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi| R/\lambda$ (for this model, $|\xi| G\phi_0^2 \ll 1$).

SM loop corrections to the tree potential leads to $\lambda = \lambda(\phi)$, then the same expression for $f(R)$ follows with

$$M^2 = \frac{\lambda(\phi(R))}{24\pi\xi^2 G}\left(1 + \mathcal{O}\left(\frac{d\ln\lambda(\phi(R))}{d\ln\phi}\right)^2\right).$$

.

The approximate shift invariance $\phi \to \phi + c,\ \ c = \mathrm{const}$ permitting slow-roll inflation for a minimally coupled inflaton scalar field transforms here to the approximate scale (dilatation) invariance
$\phi \to c\phi,\ \ R \to c^2 R,\ \ x^\mu \to x^\mu / c,\ \ \mu = 0, ..3$
in the physical (Jordan) frame. Of course, this symmetry needs not be fundamental, i.e. existing in some more microscopic model at the level of its action.

Embedding $f(R)$ gravity in supergravity

$F(\mathcal{R})$ supergravity - first constructed in S. J. Gates, Jr. and S. Ketov, Phys. Lett. B **674**, 59 (2009). The action ($8\pi G = 1$)

$$S = \int d^4x d^2\theta\, \mathcal{E} F(\mathcal{R}) + \mathrm{H.c.}$$

in a chiral 4D, $N = 1$ superspace in terms of a holomorphic function $F(\mathcal{R})$ of the covariantly-chiral scalar curvature superfield $\mathcal{R}$ and the chiral superspace density

$$\mathcal{E} = \sqrt{-g}\left(1 - 2i\theta\sigma_\alpha\bar{\psi}^\alpha + \theta^2 B\right)$$

where the chiral $N = 1$ superfield $F(\mathcal{R})$ has the scalar curvature R as the field coefficient at its θ^2-term, ψ^α is the gravitino and $B = S - iP$ is an auxiliary complex scalar non-propagating field. It is classically equivalent to the standard $N = 1$ Poincaré supergravity minimally coupled to the chiral scalar superfield via the supersymmetric Legendre-Weyl-Kähler transform.

Reduces to $f(R)$ gravity in the particular case:

$$\psi^\alpha = 0, \;\; B = 3X, \;\; \bar{X} = X.$$

The bosonic Lagrangian

$$L = 2F' \left[-\frac{R}{3} + 4X^2 \right] + 6XF \; .$$

The auxiliary field X obeys the algebraic equation of motion

$$3F + 11F'X + F'' \left[-\frac{R}{3} + 4X^2 \right] = 0$$

(here $F = F(X)$ and the prime denotes the derivative with respect to X) and can be excluded leading to $L = f(R)/2$. For $F(\mathcal{R}) = f_0 + \frac{1}{2} f_1 \mathcal{R}$ with non-vanishing and complex coefficients f_0 and f_1, the standard pure $N = 1$ supergravity with a negative cosmological term follows.

Embedding $(R + R^2)$-inflation in supergravity

$$L = -\frac{1}{2} f_1 \mathcal{R} + \frac{1}{2} f_2 \mathcal{R}^2 - \frac{1}{6} f_3 \mathcal{R}^3 + \ldots$$

with an anomalously large f_3: $f_3 \gg 1, \; f_1 \ll f_2^2 \ll f_1 f_3$.

Cubic equation for X:

$$X^3 - \frac{33 f_2}{20 f_3} X^2 - \frac{R - R_0}{30} X + \frac{f_2}{30 f_3} R = 0$$

where $R_0 = 21 f_1 / f_3 > 0$.

At the high-curvature regime $R > R_0$, $\frac{R - R_0}{R_0} \gg \left(\frac{f_2^2}{f_1 f_3} \right)^{1/3}$:

$$X^2 = \frac{R - R_0}{30}, \quad f(R) = \frac{f_1}{3} R + \frac{f_3}{180} (R - R_0)^2 .$$

Inflation occurs for $R \gg R_0$. To fit present observational data on the primordial spectrum of density perturbations in the Universe:

$$f_3 \approx 6.5 \times 10^{10} (N_{infl}/50)^2 .$$

Present DE models in $f(R)$ gravity

Much more difficult to construct. The original proposal to make $f(R)$ diverging at $R \to 0$ does not work!

An example of the viable model satisfying the first 5 viability conditions (A. A. Starobinsky, JETP Lett. **86**, 157 (2007)):

$$f(R) = R + \lambda R_0 \left(\frac{1}{\left(1 + \frac{R^2}{R_0^2}\right)^n} - 1 \right)$$

with $n \geq 2$. $f(0) = 0$ is put by hand to avoid the appearance of a cosmological constant in the flat space-time.
Similar models:
1. W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
2. A. Appleby and R. Battye, Phys. Lett. B **654**, 7 (2007).

No good microscopic justification for both the energy scale and the complicated form of $f(R)$ needed $(0 < f' < 1)$.

Recent progress in $f(R)$ gravity and cosmology

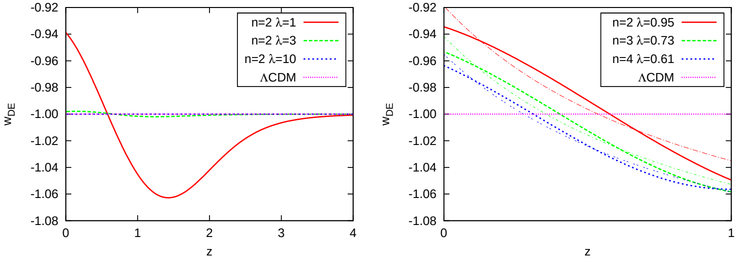
1. It was proved that viable models of DE typically exhibit phantom behaviour of dark energy during the matter-dominated stage and recent crossing of the phantom boundary $w_{\rm DE} = -1$. As a consequence of the anomalous growth of density perturbations in the cold dark matter + baryon component at recent redshifts, their growth index evolves non-monotonically with time and may even become negative temporarily (H. Motohashi, A. A. Starobinsky and J. Yokoyama, Progr. Theor. Phys. **123**, 887, 2010). Moreover, if the present mass of the scalaron is sufficiently large, there will be an infinite number of phantom boundary crossings during the future evolution of such cosmological models (H. Motohashi, A. A. Starobinsky and J. Yokoyama, JCAP **1106**, 006, 2011).

2. In order not to destroy any of previous successes of the early Universe cosmology, viable $f(R)$ models of present DE should be extended to large values of R with the $\sim R^2$ asymptotic behaviour and to negative R keeping $f'(R) > 0,\ f''(R) > 0$ at least up to the scale of inflation. Combined description of primordial and present DE using one $f(R)$ function is possible, but leads to completely different reheating after inflation during which strongly non-linear oscillations of R occur (S. A. Appleby, R. A. Battye and A. A. Starobinsky, JCAP **1006**, 005, 2010).

3. Finally, it was shown how viable $f(R)$ inflationary models can be embedded into supergravity (S. V. Ketov and A. A. Starobinsky, Phys. Rev. D **83**, 063512, 2011).

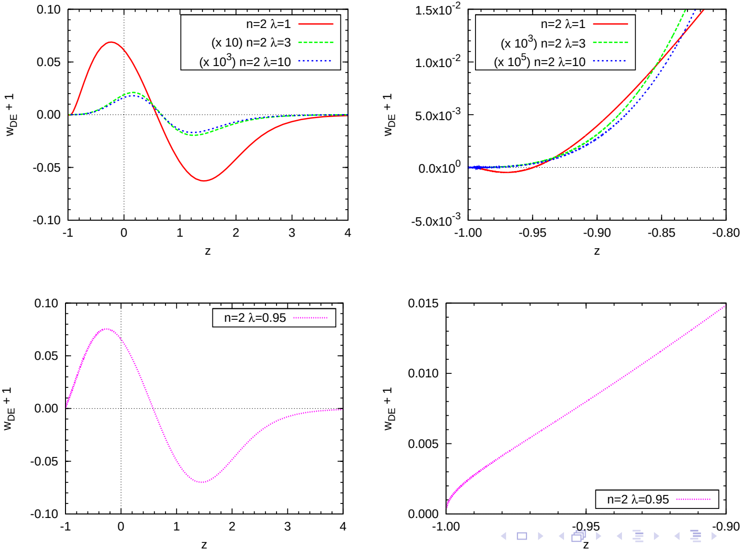
Phantom boundary crossing

Generic feature: phantom behaviour for $z > 1$,
crossing of the phantom boundary $w_{DE} = -1$ for $z < 1$.



Future evolution

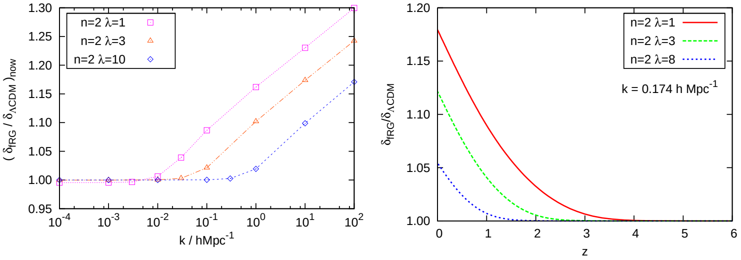
Infinite number of phantom boundary crossings at the stable future dS asymptote if $f'(R_{dS})/f''(R_{dS}) > 25R_{dS}/16$.



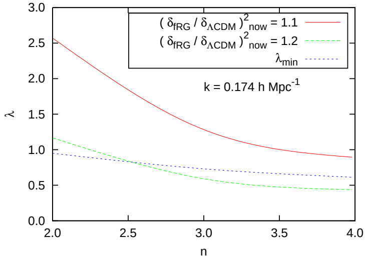
Anomalous growth of perturbations

Deeply in the sub-horizon regime:

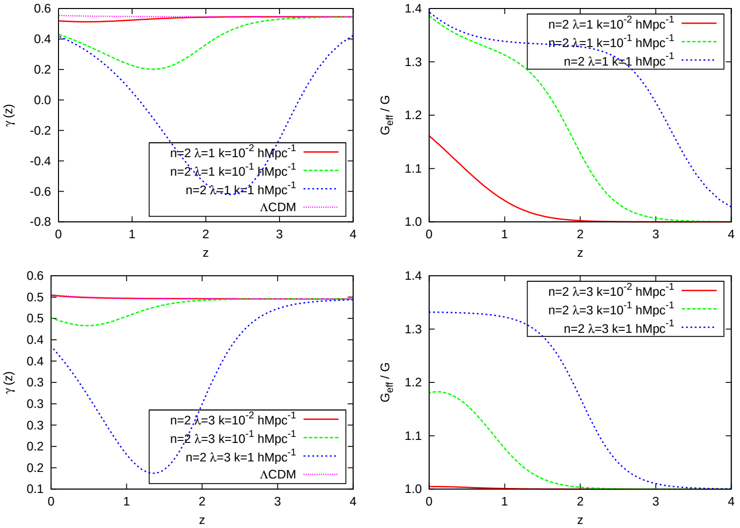
$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G_{eff}\,\rho\delta = 0\ , \quad G_{eff} = \frac{G}{f'}\,\frac{1 + 4\frac{k^2}{a^2}\frac{f''}{f'}}{1 + 3\frac{k^2}{a^2}\frac{f''}{f'}}\ .$$



Constraints in the parameter space



Evolution of $\gamma(z)$ and $G_{eff}(z)/G$



Structure of corrections to GR

$$R = R^{(0)} + \delta R_{ind} + \delta R_{osc} \ ,$$

$$R^{(0)} = 8\pi G T_m \propto a^{-3} \ ,$$

$$\delta R_{ind} = \left(RF'(R) - 2F(R) - 3\nabla_\mu \nabla^\mu F'(R)\right)_{R=R^{(0)}} \ ,$$

$$R \gg R_0, \quad \delta R_{ind} \approx \mathrm{const} = -F(\infty) = 4\Lambda(\infty) \ .$$

No Dolgov-Kawasaki instability.

$$MD: \qquad \delta R_{osc} \propto t^{-(3n+4)} \sin\left(c_1 t^{-(2n+1)} + c_2\right) ,$$

$$RD: \quad \delta R_{osc} \propto t^{-3(3n+4)/4} \sin\left(c_3 t^{-(3n+1)/2} + c_4\right) .$$

$\delta a/a$ is small but $\delta R_{osc}/R^{(0)}$ diverges for $t \to 0$.

δR_{osc} should be very small just from the beginning – a problem for those $f(R)$ models which do not let R become negative due to crossing of the $f''(R) = 0$ point.

The "scalaron overproduction" problem.

Three new problems

In the early Universe:

- Unlimited growth of $m_s(R)$ for $t \to 0$: when $m_s(R)$ exceeds M_{Pl}, quantum-gravitational loop corrections invalidate the use of an effective quasi-classical $f(R)$ gravity.
- Unlimited growth of the amplitude of δR oscillations for $t \to 0$ (the "scalaron overproduction" problem).
- "Big Boost" singularity before the Big Bang:

$$a(t) = a_0 + a_1(t - t_0) + a_2|t - t_0|^k + \ldots,\ 1 < k = \frac{4n+1}{2n+1} < 2\,,$$

if $|F(R) - F(\infty)| \propto R^{-2n}$ for $R \to \infty$, so $f''(\infty) = 0$.

Curing all three problems

S. A. Appleby, R. A. Battye and A. A. Starobinsky, JCAP **1006**, 005 (2010).

Add $\frac{R^2}{6M^2}$ to $f(R)$ with M not less than the scale of inflation. Then the first and and third problems go away. The second problem still remains, but (any) inflation can solve it. However, in all known inflationary models R may be negative during reheating after inflation (e.g. when $V(\phi) = 0$). Necessity of an extension of $f(R)$ to $R < 0$ keeping $f''(R) > 0$. As a result, a non-zero g-factor $(0 < g < 1/2)$ arises:

$$g = \frac{f'(R) - f'(-R)}{2f'(R)} , \quad R_0 \ll R \ll M^2 .$$

An example satisfying all 6 viability conditions: the g-extended R^2-corrected AB model

$$f(R) = (1-g)R + g\epsilon \log\left[\frac{\cosh\left(R/\epsilon - b\right)}{\cosh b}\right] + \frac{R^2}{6M^2} \ .$$

$m_s \approx M = const$ for $\rho_m \gg 10^{-27}$ g/cm^3 –
no "chameleon" behaviour in laboratory and Solar system experiments.

The same can be done for HSS-type models (H. Motohashi, A. A. Starobinsky and J. Yokoyama, in preparation).

Combined models of primordial and present DE

Construction of a viable model of present dark energy in $f(R)$ gravity naturally leads to combined models of primordial and present DE.

However, to take $f(R)$ simply as some function for which the equation $Rf'(R) = 2f(R)$ has 2 roots is greatly insufficient! What should be achieved in addition:
1) metastability of inflation;
2) sufficiently fast decay of the scalaron into matter quanta after inflation;
3) validity of the stability conditions $f' > 0, \ \ f'' > 0$ during all stages from inflation up to the present time.

If $M \approx 3 \times 10^{-6} M_{Pl}$, the scalaron can play the role of an inflaton, too. Then the inflationary predictions are formally the same as for the pure $R + R^2/6M^2$ inflationary model which does not describe the present DE:

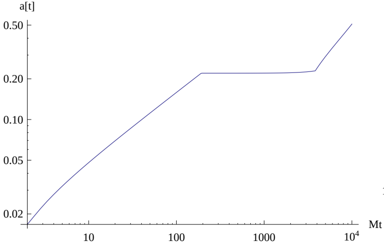
$$n_s = 1 - \frac{2}{N} \ , \quad r = \frac{12}{N^2}.$$

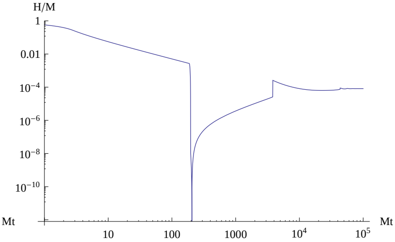
However, N is different, $N \sim 70$ for the unified model (versus $N \sim (50 - 55)$ for the purely inflationary one) because the stage if reheating after inflation becomes completely different: it consists of unequal periods with $a \approx const$ and $a \propto t^{1/2}$.
Duration of the periods in terms of $\ln t$:
$-\ln(1 - 2g)$ and $-2\ln(1 - 2g)$ respectively.
So, $a(t) \propto t^{1/3}$ on average for a long time after the end of inflation, in contrast to

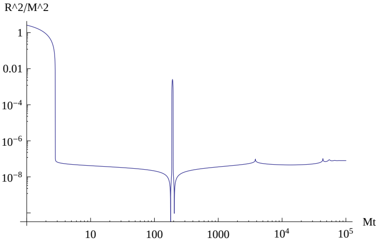
$$a(t) \propto t^{2/3} \left(1 + \frac{2}{3Mt} \sin M(t - t_1)\right)$$

for the pure inflationary $f(R) = R + R^2/6M^2$ model.

Observable prediction which is, however, degenerate with other inflationary models in $f(R)$ gravity.

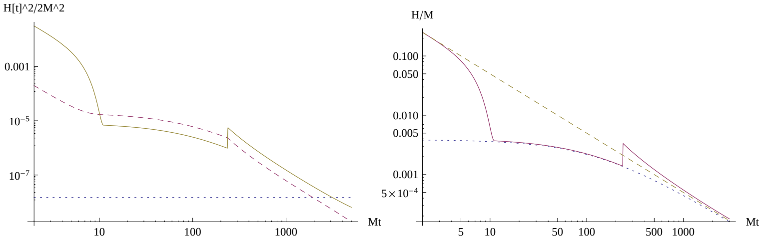

a[t]
0.50
0.20
0.10
0.05
0.02
10
100
1000
10^4
Mt


H/M
1
0.01
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10
100
1000
10^4
10^5
Mt


R^2/M^2
1
0.01
10^{-4}
10^{-6}
10^{-8}
10
100
1000
10^4
10^5
Mt

Reheating – due to gravitational particle creation which occurs mainly at the end of inflation. Less efficient than in the pure inflationary $f(R) = R + R^2/6M^2$ model,

$$t = t_{\rm reh} \sim M^{-4} M_{\rm Pl}^3 \sim 10^{-18}~{\rm s}~.$$



Conclusions

- Modified gravity describing primordial or present dark energy can supply a new scalar degree of freedom (scalar particle) universally interacting with all other matter. The simplest of purely geometrical DE is provided by $f(R)$ gravity. However, generically there is no strict border between physical DE (a new semi-fundamental scalar field) and geometrical DE (modified gravity). Even some models of $f(R)$ gravity may arise from a non-minimally coupled scalar field in some limit.
- The simplest pioneer inflationary $f(R)$ model remains viable. The critical test: the low value for the tensor-to-scalar ratio of primordial metric perturbations $r \sim 0.4\%$. This model can be embedded in supergravity.

- Much more problems with models of present DE. Still a narrow class among all $f(R)$ models of present DE remains viable: it is possible to construct predictive models satisfying all existing cosmological, Solar system and laboratory data, and distinguishable from ΛCDM. However, these models require a complicated structure of $f(R)$ at low R for which no simple microscopic explanation is known at present.
- In order not to destroy all previous successes of the early Universe cosmology, these viable $f(R)$ models of present DE should be extended to large R with the $\sim R^2$ asymptotic behaviour and to negative R keeping $f'(R) > 0, \ f''(R) > 0$ at least up to the scale of inflation.

- This naturally (though not inevitably) leads to combined models of primordial and present DE for the specific choice of M: $M \approx 3 \times 10^{-6} M_{Pl}$.
- Combined inflationary – DE $f(R)$ models have a significantly different reheating stage after inflation as compared to pure inflationary $f(R)$ models, with strongly non-linear oscillations of the scale factor $a(t)$. The ultimate reason for this: different values of G_{eff} for $R > 0$ and $R < 0$ due to $f''(R) > 0$.
- The most critical test for all $f(R)$ models of present dark energy: anomalous growth of density perturbations in the matter component at recent redshifts $z \sim 1 - 3$. A number of different ways to check it in the linear and non-linear regimes.